REDUCTION OF FILTERED K-THEORY AND
A CHARACTERIZATION OF CUNTZ-KRIEGER ALGEBRAS

SARA ARKLINT, RASMUS BENTMANN, AND TAKESHI KATSURA

ABSTRACT. We show that filtered K-theory is equivalent to a substantially smaller invariant for all real-rank-zero C^*-algebras with certain primitive ideal spaces – including the infinitely many so-called accordion spaces for which filtered K-theory is known to be a complete invariant. We also determine the range of filtered K-theory for purely infinite graph algebras with these primitive ideal spaces. As a consequence, we give a characterization of purely infinite Cuntz-Krieger algebras with accordion spaces as primitive ideal spaces.

1. INTRODUCTION

The Cuntz and Cuntz-Krieger algebras are historically and in general of great importance for our understanding of simple and non-simple purely infinite C^*-algebras as they were not only the first constructed examples of such but are also very tangible due to the combinatorial nature of their construction, [12]. The Cuntz-Krieger algebras arise from shifts of finite type, and it has been shown that they are exactly the graph algebras $C^*(E)$ arising from finite directed graphs E with no sources, [3]. The Cuntz algebras and the simple Cuntz-Krieger algebras can be identified as the UCT Kirchberg algebras with a specific type of K-theory. A similar characterization for non-simple, purely infinite Cuntz-Krieger algebras and, more generally, of unital graph algebras of this type is desirable.

A Kirchberg X-algebra is a purely infinite, nuclear, separable C^*-algebras with primitive ideal space X. When X is a so-called accordion space, cf. Definition 2.2, the invariant *filtered K-theory* FK is a strongly complete invariant for stable Kirchberg X-algebras with simple subquotients in the boostrap class [5,16,18]. In particular filtered K-theory is complete for purely infinite graph algebras with accordion spaces as primitive ideal spaces, and the main goal of this paper is to use this to achieve a *characterization* of purely infinite Cuntz-Krieger algebras and graph algebras.

In this paper we determine the range of *reduced* filtered K-theory $FK_{\mathcal{R}}$ with respect to purely infinite Cuntz-Krieger algebras and graph algebras. The reduced filtered K-theory was originally defined by Gunnar Restorff, [22], who used it to give an internal classification of the purely infinite Cuntz-Krieger algebras, inspired by work of Mikael Rørdam, [24], and work of Mike Boyle and Danrun Huang on

dynamical systems, [8]. We also show that for purely infinite graph algebras whose primitive ideal spaces are of a certain type including the accordion spaces, reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ is equivalent to (concrete) filtered K-theory. This is done by introducing the filtered K-theory *restricted to the canonical base*, denoted $\mathrm{FK}_{\mathcal{B}}$, and showing that it is equivalent to the filtered K-theory for real-rank-zero C^*-algebras whose primitive ideal spaces are of a certain type including the accordion spaces.

For accordion spaces we thereby obtain one-to-one correspondences, induced by these filtered K-theories, between purely infinite graph algebras and certain types of modules in the target categories of the corresponding filtered K-theory, cf. Theorem 11.1 and its corollaries. In particular we obtain the desired characterization of purely infinite Cuntz-Krieger algebras with accordion spaces as primitive ideal spaces:

Theorem 1.1. *Let A be a C^*-algebra whose primitive ideal space is an accordion space. Then A is a purely infinite Cuntz-Krieger algebra if and only if A satisfies the following:*

- *A is unital, purely infinite, nuclear, separable, and of real rank zero,*
- *for all ideals I and J of A with $I \subseteq J$ and J/I simple, the quotient J/I is in the bootstrap class and satisfies that $\mathrm{K}_*(J/I)$ is finitely generated, that $\mathrm{K}_1(J/I)$ is free and that $\mathrm{rank}\,\mathrm{K}_1(J/I) = \mathrm{rank}\,\mathrm{K}_0(J/I)$.*

In the terms introduced by the first named author in [1], our Theorem 1.1 states that there are no *phantom Cuntz-Krieger algebras* with accordion spaces as primitive ideal spaces. It is an important open question whether this holds for all finite primitive ideal spaces.

1.1. **Historical account.** By a seminal result of Eberhard Kirchberg, $\mathrm{KK}(X)$-equivalences between stable Kirchberg X-algebras, i.e., stable, tight, $\mathcal{O}_\infty$-absorbing, nuclear, separable C^*-algebras over a space X, lift to X-equivariant *-isomorphisms. With the aim of computing the equivariant bivariant theory $\mathrm{KK}(X)$, Ralf Meyer and Ryszard Nest established in [18] a Universal Coefficient Theorem for filtered K-theory FK over any finite totally ordered space X. As a result, for such spaces X, isomorphisms on filtered K-theory between stable Kirchberg X-algebras with simple subquotients in the bootstrap class lift to X-equivariant *-isomorphisms. This result was generalized in [4, 5] by the second named author and Manuel Köhler to the case of so-called accordion spaces. Building on these results, Søren Eilers, Gunnar Restorff, and Efren Ruiz classified in [14] certain classes of real-rank-zero (not necessarily purely infinite) graph algebras using *ordered* filtered K-theory.

On the other hand, Meyer-Nest and the second named author constructed counterexamples to classification for all six four-point non-accordion spaces. More precisely, for each such X they find two non-$\mathrm{KK}(X)$-equivalent Kirchberg X-algebras with simple subquotients in the bootstrap class whose filtered K-theories are isomorphic (see [4, 18]).

Despite this obstruction, it had previously been shown by Gunnar Restorff in [22] that filtered K-theory FK, and in fact the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$, is a complete invariant for purely infinite Cuntz-Krieger algebras. Any finite T_0-space, in particular the six problematic four-point spaces, can be realized as the primitive ideal space of a purely infinite Cuntz-Krieger algebra. Unfortunately, Restorff's result only gives an *internal* classification of Cuntz-Krieger algebras and admits no conclusion concerning when a given Cuntz-Krieger algebra is stably isomorphic to

a given purely infinite, nuclear, separable C^*-algebra with the same ideal structure and filtered K-theory.

The purely infinite Cuntz-Krieger algebras have real rank zero. In [2], Gunnar Restorff, Efren Ruiz, and the first named author noted that for five of the six problematic four-point spaces the constructed counterexamples to classification do *not* have real rank zero. They went on to show that for four of these spaces X, filtered K-theory is in fact a complete invariant for Kirchberg X-algebras of real rank zero with simple subquotients in the bootstrap class. The four-point non-accordion space for which the constructed counterexample has real rank zero will be denoted by $\mathcal{D}$.

For a Cuntz-Krieger algebra the K_1-group of every subquotient is free. The same is true, more generally, for graph algebras. We observe that, for real rank zero C^*-algebras over $\mathcal{D}$ satisfying this condition on their K-theory, isomorphisms on the reduced filtered K-theory $FK_{\mathcal{R}}$ lift to $KK(\mathcal{D})$-equivalences (see Proposition 7.16). There are therefore no known counterexamples to classification by filtered K-theory of Kirchberg X-algebras with simple subquotients in the bootstrap class that have the K-theory of a real rank zero graph algebra.

1.2. Organization of the paper.

The main focus of this paper is not completeness of the invariant filtered K-theory, but *reduction* of filtered K-theory under the assumption of real rank zero and determining the range of filtered K-theory for purely infinite graph algebras. The main results are recaptured in Theorem 11.1 and its corollaries.

Filtered K-theory FK and concrete filtered K-theory $FK_{\mathcal{ST}}$ are defined in Section 3. In Section 5, filtered K-theory restricted to a canonical base $FK_{\mathcal{B}}$ is defined for spaces with the unique path property, and it is shown for certain spaces that the concrete filtered K-theory $FK_{\mathcal{ST}}(A)$ of a real rank zero C^*-algebra A is completely determined by the filtered K-theory restricted to a canonical base $FK_{\mathcal{B}}(A)$, cf. Corollary 5.19.

In Section 6, reduced filtered K-theory $FK_{\mathcal{R}}$ is defined, and it is shown in Section 7 for certain spaces that the concrete filtered K-theory $FK_{\mathcal{ST}}(A)$ of a real rank zero C^*-algebra A satisfying that all subquotients have free K_1-groups can be recovered from the reduced filtered K-theory $FK_{\mathcal{R}}(A)$, cf. Corollary 7.15. This is of particular interest since in Section 8 (and 9) we determine the range of (unital) reduced filtered K-theory $FK_{\mathcal{R}}$ for (unital) purely infinite graph algebras, cf. Theorem 8.2 (and 9.10).

In Sections 9 and 10, unital filtered K-theories and ordered filtered K-theories are treated.

1.3. Acknowledgements.

2. Notation

We follow the notation and definition for graph algebras of Iain Raeburn, cf. [21]. All graphs are assumed to be countable and to satisfy Condition (K), hence all considered graph algebras are separable and of real rank zero. In this article, matrices act from the right and the composite of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is denoted by fg. The category of abelian groups is denoted by $\mathfrak{Ab}$, the category of $\mathbb{Z}/2$-graded abelian groups by $\mathfrak{Ab}^{\mathbb{Z}/2}$.

Let X be a finite T_0-space. For a subset Y of X, we let $\overline{Y}$ denote the closure of Y in X, and let $\overline{\partial}Y$ denote the closed boundary $\overline{Y} \setminus Y$ of Y. Since X is a finite space, there exists a smallest open subset $\widetilde{Y}$ of X containing Y. We let $\widetilde{\partial}Y$ denote the set $\widetilde{Y} \setminus Y$.

For $x, y \in X$ we write $x \leq y$ when $\overline{\{x\}} \subseteq \overline{\{y\}}$, and $x < y$ when $x \leq y$ and $x \neq y$. We write $y \to x$ when $x < y$ and no $z \in X$ satisfies $x < z < y$. The following lemma is straightforward to verify.

Lemma 2.1. *For $x \in X$, the following hold:*

(1) *An element $y \in X$ satisfies $y \to x$ if and only if y is a closed point of $\widetilde{\partial}\{x\}$.*
(2) *We have $\widetilde{\partial}\{x\} = \bigcup_{y \to x} \widetilde{\{y\}}$, and consequently $\widetilde{\partial}\{x\}$ is open.*
(3) *An element $y \in X$ satisfies $x \leq y$ if and only if there exists a finite sequence $(z_k)_{k=1}^{n}$ in X such that $z_{k+1} \to z_k$ for $k = 1, \ldots, n-1$ where $z_1 = x$, $z_n = y$.*

We call a sequence $(z_k)_{k=1}^{n}$ as in Lemma 2.1(3) a *path* from y to x. We denote by $\mathrm{Path}(y, x)$ the set of paths from y to x. Thus Lemma 2.1(3) can be rephrased as follows: two points $x, y \in X$ satisfy $x \leq y$ if and only if there exists a path from y to x. Such a path is unique if X is an accordion space, but not in general. Two points $x, y \in X$ satisfy $y \to x$ if and only if (x, y) is a path from y to x, and in this case, there are no other paths.

Definition 2.2. An *accordion space* is a T_0-space $X = \{x_1, \ldots, x_n\}$ such that for every $k = 1, 2, \ldots, n-1$ either $x_k \to x_{k+1}$ or $x_k \leftarrow x_{k+1}$ holds and such that $x_k \to x_l$ does not hold for any k, l with $|k - l| \neq 1$.

If X is linear, i.e., if $X = \{x_1, \ldots, x_n\}$ with $x_n \to \cdots \to x_2 \to x_1$, then X is an accordion space.

3. Filtered K-theory

In this section filtered K-theory and concrete filtered K-theory together with target categories are defined. Various properties of objects in the target categories are introduced.

A C^*-*algebra A over X* is a C^*-algebra A equipped with an monotone, infima- and suprema-preserving map $\mathbb{O}(X) \to \mathbb{I}(A), U \mapsto A(U)$ mapping open subsets in X to ideals in A. The C^*-algebra A is called *tight* over X if the map is a lattice-isomorphism. A *-homomorphism $\varphi \colon A \to B$ for C^*-algebras A and B over X is called X-*equivariant* if $\varphi(A(U)) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$. Let $\mathbb{LC}(X)$ denote the set of locally closed subsets of X, i.e., subsets of the form $U \setminus V$ with U and V open subsets of X satisfying $V \subseteq U$. For $Y \in \mathbb{LC}(X)$, and $U, V \in \mathbb{O}(X)$ satisfying that $Y = U \setminus V$ and $U \supseteq V$, we define $A(Y)$ as $A(Y) = A(U)/A(V)$, which up to natural isomorphism is independent of the choice of U and V (see [19, Lemma 2.15]).

Definition 3.1. A tight, O_∞-absorbing, nuclear, separable C^*-algebra over X is called a *Kirchberg X-algebra*.

Let $\mathfrak{KK}(X)$ be the additive category whose objects are separable C^*-algebras over X and whose set of morphisms from A to B is the Kasparov group $\mathrm{KK}_0(X; A, B)$ defined by Kirchberg (see [19, Section 3] for details). For a C^*-algebra A over X, a $\mathbb{Z}/2$-graded abelian group $\mathrm{FK}_Y^*(A)$ is defined as $\mathrm{K}_*(A(Y))$ for all $Y \in \mathbb{LC}(X)$. Thus FK_Y^* is an additive funtor from $\mathfrak{KK}(X)$ to the category $\mathfrak{Ab}^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$-graded abelian groups. Ralf Meyer and Ryszard Nest constructed in [18] C^*-algebras R_Y over X satisfying that the functors FK_Y^* and $\mathrm{KK}_*(X; R_Y, -)$ are naturally isomorphic.

In their definition of filtered K-theory FK^*, Meyer-Nest consider the $\mathbb{Z}/2$-graded preadditive category $\mathcal{NT}_*$ with objects $\mathbb{LC}(X)$ and morphisms

$$\mathrm{Nat}_*(\mathrm{FK}_Y^*, \mathrm{FK}_Z^*) \cong \mathrm{KK}_*(X; R_Z, R_Y)$$

between Y and Z, where $\mathrm{Nat}_*(\mathrm{FK}_Y^*, \mathrm{FK}_Z^*)$ denotes the set of graded natural transformations from the functor FK_Y^* to the functor FK_Z^*. The target category of FK^* is the category $\mathfrak{Mod}(\mathcal{NT}_*)^{\mathbb{Z}/2}$ of graded modules over $\mathcal{NT}_*$, that is, $\mathbb{Z}/2$-graded additive functors $\mathcal{NT}_* \to \mathfrak{Ab}^{\mathbb{Z}/2}$. Hence $\mathrm{FK}^*(A)$ consists of the groups $\mathrm{FK}_Y^*(A)$ together with the natural transformations $\mathrm{FK}_Y^*(A) \to \mathrm{FK}_Z(A)^*$.

For reasons of notation we will often find it convenient to consider instead the preadditive category $\mathcal{NT}$ with objects $\mathbb{LC}(X) \times \{0, 1\}$ and morphisms between (Y, j) and (Z, k) given by natural transformations

$$\mathrm{Nat}(\mathrm{FK}_Y^j \, \mathrm{FK}_Z^k) \cong \mathrm{KK}_0(X; \mathrm{S}^k \, R_Z, \mathrm{S}^j \, R_Y),$$

where $\mathrm{FK}_Y^j(A)$ denotes $\mathrm{K}_j(A(Y))$ for $j = 0, 1$. Let $\mathfrak{Mod}(\mathcal{NT})$ denote the category of modules over $\mathcal{NT}$, that is, additive functors $\mathcal{NT} \to \mathfrak{Ab}$.

Given a graded $\mathcal{NT}_*$-module M, we define an $\mathcal{NT}$-module $D(M)$ as follows: we set $D(M)(Y, i) = M(Y)_i$ for $(Y, i) \in \mathbb{LC}(X) \times \{0, 1\}$; for a morphism $f \colon (Y, i) \to (Z, j)$ in $\mathcal{NT}$, we define $D(M)(f) \colon D(M)(Y, i) \to D(M)(Z, j)$ as the composite

$$M(Y)_i \hookrightarrow M(Y)_* \xrightarrow{M(f)} M(Z)_* \twoheadrightarrow M(Z)_j.$$

It is straightforward to check that this yields a functor $D \colon \mathfrak{Mod}(\mathcal{NT}_*)^{\mathbb{Z}/2} \to \mathfrak{Mod}(\mathcal{NT})$. In fact, D is an equivalence of categories—an inverse can be defined by a direct sum construction. Consequently, we define the functor $\mathrm{FK} \colon \mathfrak{KK}(X) \to \mathfrak{Mod}(\mathcal{NT})$ as the composite $\mathrm{FK} = D \circ \mathrm{FK}^*$.

Definition 3.2. Let $Y \in \mathbb{LC}(X)$, $U \subseteq Y$ be open and set $C = Y \setminus U$. A pair (U, C) obtained in this way is called a *boundary pair*. The natural transformations occuring in the six-term exact sequence in K-theory for the distinguished subquotient inclusion associated to $U \subseteq Y$ are denoted by i_U^Y, r_Y^C and δ_C^Y:

$$
\begin{array}{ccc}
\mathrm{FK}_U & \xrightarrow{\;\;i_U^Z\;\;} & \mathrm{FK}_Y \\
& & \\
{\scriptstyle \delta_C^U} \nwarrow & & \swarrow {\scriptstyle r_Y^C} \\
& \mathrm{FK}_C &
\end{array}
$$

These elements i_U^Y, r_Y^C and δ_C^Y correspond to the $\mathrm{KK}(X)$-classes of the *-homomorphisms $R_Y \twoheadrightarrow R_U$, $R_C \hookrightarrow R_Y$, and the extention $R_C \hookrightarrow R_Y \twoheadrightarrow R_U$, cf. [18]. In $\mathcal{NT}_*$ they satisfy the following relations.

Proposition 3.3. *In the category $\mathcal{NT}_*$, the following relations hold.*

(1) *For every $Y \in \mathbb{LC}(X)$,*

$$i_Y^Y = r_Y^Y = \mathrm{id}_Y .$$

(2) *If $Y \sqcup Z$ is a topologically disjoint union of $Y, Z \in \mathbb{LC}(X)$, then*

$$r_{Y \cup Z}^Y i_Y^{Y \cup Z} + r_{Y \cup Z}^Z i_Z^{Y \cup Z} = \mathrm{id}_{Y \cup Z} .$$

(3) *For $Y \in \mathbb{LC}(X)$ and open subsets $U \subseteq V \subseteq Y$,*

$$i_U^V i_V^Y = i_U^Y.$$

(4) *For $Y \in \mathbb{LC}(X)$ and closed subsets $C \subseteq D \subseteq Y$,*

$$r_Y^D r_D^C = r_Y^C.$$

(5) *For $Y \in \mathbb{LC}(X)$, an open subset $U \subseteq Y$ and a closed subset $C \subseteq Y$,*

$$i_U^Y r_Y^C = r_U^{U \cap C} i_{U \cap C}^C.$$

(6) *For a boundary pair (U, C) in X and an open subset $C' \subseteq C$, (U, C') is a boundary pair and we have*

$$i_{C'}^C \delta_C^U = \delta_{C'}^U.$$

(7) *For a boundary pair (U, C) in X and a closed subset $U' \subseteq U$, (U', C) is a boundary pair and we have*

$$\delta_C^U r_U^{U'} = \delta_C^{U'} .$$

(8) *For $Y, Z, W \in \mathbb{LC}(X)$ such that $Y \cup W \in \mathbb{LC}(X)$ containing Y, W as closed subsets, $Z \cup W \in \mathbb{LC}(X)$ containing Z, W as open subsets, and $W \subseteq Y \cup Z$, we have*

$$\delta_Y^{W \setminus Y} i_{W \setminus Y}^Z = r_Y^{W \setminus Z} \delta_{W \setminus Z}^Z.$$

Proof. We only prove (8), because the other relations can be proved similarly and more easily (their proofs can be found in [4, Section 3.2]).

Let us take $Y, Z, W \in \mathbb{LC}(X)$ as in (8). Let us also take a C^*-algebra A over X. Since both Y and W are closed subsets of $Y \cup W \in \mathbb{LC}(X)$, $Y \cap W$ is closed both in Y and in W. Therefore we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A(W \setminus Y) & \longrightarrow & A(Y \cup W) & \longrightarrow & A(Y) & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A(W \setminus Y) & \longrightarrow & A(W) & \longrightarrow & A(Y \cap W) & \longrightarrow & 0.
\end{array}
$$

Since both Z and W are open subsets of $Z \cup W \in \mathbb{LC}(X)$, $Z \cap W$ is open both in Z and in W. Therefore we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A(Z \cap W) & \longrightarrow & A(W) & \longrightarrow & A(W \setminus Z) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A(Z) & \longrightarrow & A(Z \cup W) & \longrightarrow & A(W \setminus Z) & \longrightarrow & 0.
\end{array}
$$

From $W \subseteq Y \cup Z$, we get $W \setminus Y \subseteq Z \cap W$ and $W \setminus Z \subseteq Y \cap W$. Since $W \setminus Y$ is open in W, we see that $W \setminus Y$ is open in $Z \cap W$. Similarly, $W \setminus Z$ is closed in $Y \cap W$. Hence we get a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A(W \setminus Y) & \longrightarrow & A(W) & \longrightarrow & A(Y \cap W) & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & A(Z \cap W) & \longrightarrow & A(W) & \longrightarrow & A(W \setminus Z) & \longrightarrow & 0.
\end{array}
$$

By combining these three diagrams, we obtain a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A(W \setminus Y) & \longrightarrow & A(Y \cup W) & \longrightarrow & A(Y) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A(Z) & \longrightarrow & A(Z \cup W) & \longrightarrow & A(W \setminus Z) & \longrightarrow & 0.
\end{array}
$$

From this digram, we get a commutative diagram

$$
\begin{array}{ccccccccc}
\mathrm{K}_*(A(Y \cup W)) & \xrightarrow{r} & \mathrm{K}_*(A(Y)) & \xrightarrow{\delta} & \mathrm{K}_*(A(W \setminus Y)) & \xrightarrow{i} & \mathrm{K}_*(A(Y \cup W)) \\
\downarrow{\scriptstyle ri} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle ri} \\
\mathrm{K}_*(A(Z \cup W)) & \xrightarrow{r} & \mathrm{K}_*(A(W \setminus Z)) & \xrightarrow{\delta} & \mathrm{K}_*(A(Z)) & \xrightarrow{i} & \mathrm{K}_*(A(Z \cup W)).
\end{array}
$$

Now (8) follows from the commutativity of the middle square of this natural diagram. $\qquad\square$

Remark 3.4. From Proposition 3.3(2), we see that the empty set $\emptyset$ is a zero object in $\mathcal{NT}_*$. From this and other relations in Proposition 3.3, we see the vanishing of consecutive maps in six-term sequences associated to distinguished subquotient inclusions.

Remark 3.5. The even and odd components of the element i_U^Y in $\mathcal{NT}_*$ defined in Definition 3.2 are denoted by $(i_U^Y, 0)$ and $(i_U^Y, 1)$ (or $i_{(U,0)}^{(Y,0)}$ and $i_{(U,1)}^{(Y,1)}$) respectively. These are morphisms in the category $\mathcal{NT}$. If the meaning is clear from the context, both elements are denoted simply by i. Similar comments apply to r and δ.

Definition 3.6. Let $\mathcal{ST}_*$ be the universal $\mathbb{Z}/2$-graded preadditive category whose set of objects is $\mathbb{LC}(X)$ and whose set of morphisms are generated by elements as in Definition 3.2 with the relations as in Proposition 3.3.

Let $\mathcal{ST}$ be the preadditive category whose set of objects is $\mathbb{LC}(X) \times \{0,1\}$ and which is naturally equivalent to $\mathcal{ST}_*$.

By Proposition 3.3, we have a canonical additive functor $\mathcal{ST} \to \mathcal{NT}$. This functor has been shown to be an isomorphism in all examples which have been investigated—including accordion spaces and all four-point spaces (see [4,18]). However there is an example Q of a finite T_0 space for which the functor $\mathcal{ST} \to \mathcal{NT}$ seems not to be faithful (see Remark 5.16). For such spaces we need to modify the definition of $\mathcal{ST}$, but we do not pursue this problem in this paper.

Let $\mathfrak{F}_{\mathcal{ST}} \colon \mathfrak{Mod}(\mathcal{NT}) \to \mathfrak{Mod}(\mathcal{ST})$ be the functor induced by the canonical functor $\mathcal{ST} \to \mathcal{NT}$.

Definition 3.7. We define *concrete filtered* K-*theory* $\mathrm{FK}_{\mathcal{ST}} \colon \mathfrak{KK}(X) \to \mathfrak{Mod}(\mathcal{ST})$ as the composition $\mathfrak{F}_{\mathcal{ST}} \circ \mathrm{FK}$.

Remark 3.8. As noted above, filtered K-theory FK and concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}$ coincide for accordion spaces and all four-point spaces.

Definition 3.9. An $\mathcal{NT}$-module M is called *exact* if for all $Y \in \mathbb{LC}(X)$ and $U \in \mathbb{O}(Y)$, the sequence

$$
\begin{array}{ccccc}
M(U,0) & \xrightarrow{\ i\ } & M(Y,0) & \xrightarrow{\ r\ } & M(Y \setminus U,0) \\
{\scriptstyle\delta}\big\uparrow & & & & \big\downarrow{\scriptstyle\delta} \\
M(Y \setminus U,1) & \xleftarrow{\ r\ } & M(Y,1) & \xleftarrow{\ i\ } & M(U,1)
\end{array}
$$

is exact. An $\mathcal{NT}$-module M is called *real-rank-zero-like* if for all $Y \in \mathbb{LC}(X)$ and $U \in \mathbb{O}(Y)$, the map $\delta \colon M(Y \setminus U,0) \to M(U,1)$ vanishes.

In the same way, we define exact $\mathcal{ST}$-modules and real-rank-zero-like $\mathcal{ST}$-modules.

Remark 3.10. For a C^*-algebra A over X, the module $\mathrm{FK}(A)$ is exact. It follows from [6, Lemma 3.4] that, if A is tight over X, then $\mathrm{FK}(A)$ is real-rank-zero-like if and only if the underlying C^*-algebra of A is K_0-*liftable* in the sense of Pasnicu-Rørdam [20]. By [17, Proposition 4], all real-rank-zero C^*-algebras are K_0-liftable. By Theorem 4.2 and Example 4.8 of [20], a tight, purely infinite C^*-algebra A over X has real rank zero if and only if $\mathrm{FK}(A)$ is real-rank-zero-like. Analogous remarks apply with $\mathrm{FK}_{\mathcal{ST}}(A)$ in place of $\mathrm{FK}(A)$.

Theorem 3.11 ([5,16,18]). *Let X be an accordion space, and let A and B be stable Kirchberg X-algebras with all simple subquotients in the bootstrap class. Then any isomorphism $\mathrm{FK}(A) \to \mathrm{FK}(B)$ lifts to an isomorphism $A \to B$.*

4. Sheaves

In this section we introduce sheaves and cosheaves and recall that it suffices to specify them on a basis for the topology.

Let X be an arbitrary topological space. Let $\mathbb{B}$ be a basis for the topology on X. We note that the set $\mathbb{O}$ of all open subsets is the largest basis for the topology on X. We also note that for a finite space X, the collection $\{\widetilde{\{x\}} \mid x \in X\}$ is an example of a basis. The set $\mathbb{B}$ is a category whose morphisms are inclusions.

Definition 4.1. A *presheaf on $\mathbb{B}$* is a contravariant functor $M \colon \mathbb{B} \to \mathfrak{Ab}$. It is a *sheaf on $\mathbb{B}$* if, for every $U \in \mathbb{B}$, every covering $\{U_j\}_{j \in J} \subseteq \mathbb{B}$ of U, and all coverings $\{U_{jkl}\}_{l \in L_{jk}} \subseteq \mathbb{B}$ of $U_j \cap U_k$, the sequence

$$
(4.2) \quad 0 \longrightarrow M(U) \xrightarrow{\left(M(i_U^{U_j})\right)} \prod_{j \in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U_{jkl}}) - M(i_{U_k}^{U_{jkl}})\right)} \prod_{j,k \in J} \prod_{l \in L_{jk}} M(U_{jkl})
$$

is exact. A morphism for sheaves is a natural transformation of functors. We denote by $\mathfrak{Sh}(\mathbb{B})$ the category of sheaves on $\mathbb{B}$.

If $\mathbb{B}$ is closed under intersection (for example if $\mathbb{B} = \mathbb{O}$), then the definition of sheaf can be replaced with the exactness of the sequence

$$
0 \longrightarrow M(U) \xrightarrow{\left(M(i_U^{U_j})\right)} \prod_{j \in J} M(U_j) \xrightarrow{\left(M(i_{U_j}^{U_j \cap U_k}) - M(i_{U_k}^{U_j \cap U_k})\right)} \prod_{j,k \in J} M(U_j \cap U_k)
$$

for all $U \in \mathbb{B}$ and every covering $\{U_j\}_{j \in J} \subseteq \mathbb{B}$ of U.

Lemma 4.3. *For a basis $\mathbb{B}$ for the topology on X, the restriction functor $\mathfrak{Sh}(\mathbb{O}) \to \mathfrak{Sh}(\mathbb{B})$ is an equivalence of categories.*

Proof. This is a well-known fact in algebraic geometry (see, for instance the encyclopedic treatment in [25, Lemma 009O]). We confine ourselves on mentioning that (4.2) provides a formula for computing $M(U)$ for an arbitrary open subset U. □

Definition 4.4. A *precosheaf on $\mathbb{B}$* is a covariant functor $M \colon \mathbb{B} \to \mathfrak{Ab}$. It is a *cosheaf on $\mathbb{B}$* if, for every $U \in \mathbb{B}$, every covering $\{U_j\}_{j \in J} \subseteq \mathbb{B}$ of U, and all coverings $\{U_{jkl}\}_{l \in L_{jk}} \subseteq \mathbb{B}$ of $U_j \cap U_k$, the sequence

$$(4.5) \qquad \bigoplus_{j,k \in J} \bigoplus_{l \in L_{jk}} M(U_{jkl}) \xrightarrow{\left(M(i^{U_j}_{U_{jkl}}) - M(i^{U_k}_{U_{jkl}})\right)} \bigoplus_{j \in J} M(U_j) \xrightarrow{\left(M(i^{U}_{U_j})\right)} M(U) \longrightarrow 0.$$

is exact. A morphism for cosheaves is a natural transformation of functors. We denote by $\mathfrak{CoSh}(\mathbb{B})$ the category of cosheaves on $\mathbb{B}$.

Similarly to the case of sheaves, if $\mathbb{B}$ is closed under intersection, the definition of cosheaf can be replaced with the exactness of the sequence

(4.6)

$$\bigoplus_{j,k \in J} M(U_j \cap U_k) \xrightarrow{\left(M(i^{U_j}_{U_j \cap U_k}) - M(i^{U_k}_{U_j \cap U_k})\right)} \bigoplus_{j \in J} M(U_j) \xrightarrow{\left(M(i^{U}_{U_j})\right)} M(U) \longrightarrow 0.$$

for $U \in \mathbb{B}$ and a covering $\{U_j\}_{j \in J} \subseteq \mathbb{B}$ of U.

Lemma 4.7. *The restriction functor $\mathfrak{CoSh}(\mathbb{O}) \to \mathfrak{CoSh}(\mathbb{B})$ is an equivalence of categories.*

Proof. This statement is the dual of Lemma 4.3 and follows in an analogous way. Again, (4.5) can be used to compute $M(U)$ for an arbitrary open subset U. □

With regard to the next section we remark that every finite T_0-space (more generally every Alexandrov space) comes with canonical bases for the open subsets, namely $\left\{\widetilde{\{x\}} \mid x \in X\right\}$, and for the closed subsets: $\left\{\overline{\{x\}} \mid x \in X\right\}$.

Lemma 4.8. *Let X be a finite T_0-space and let S be a pre(co)sheaf on the basis $\mathbb{B} = \left\{\widetilde{\{x\}} \mid x \in X\right\}$. Then S is a (co)sheaf.*

Proof. This follows from the observation that in the basis $\mathbb{B}$ there are no non-trivial coverings, that is, if $\mathcal{U}$ is a covering of U, then $U \in \mathcal{U}$. □

5. Filtered K-theory restricted to the canonical base

In this section, the functor $\mathrm{FK}_{\mathcal{B}}$ and the notions of UP spaces and EBP spaces are introduced. The following lemma is straightforward to verify.

Lemma 5.1. *For a finite T_0-space X the following conditions are equivalent.*
- *For all $x, y \in X$, there is at most one path from y to x.*
- *There are no elements a, b, c, d in X with $a < b < d$, $a < c < d$ and neither $b \leq c$ nor $c \leq b$.*
- *For all $x, y \in X$ with $x \to y$, we have $\widetilde{\{x\}} \cup \overline{\{y\}} \in \mathbb{LC}(X)$.*

- *For every boundary pair (U, C), the pair $(\widetilde{U}, \overline{C})$ is a boundary pair.*
- *For all $x \in X$, $\widetilde{\partial}\{x\} = \coprod_{y \to x} \widetilde{\{y\}}$.*
- *For all $x \in X$, $\overline{\partial}\{x\} = \coprod_{y \leftarrow x} \overline{\{y\}}$.*

Definition 5.2. A finite T_0-space X is called *UP* (*unique path*) if it satisfies the equivalent conditions specified in Lemma 5.1.

Let X be a UP space.

Definition 5.3. Let $\mathcal{B}$ denote the universal preadditive category generated by objects $\overline{x}_1$, $\widetilde{x}_0$ for all $x \in X$ and morphisms $r^{\overline{y}_1}_{\overline{x}_1}$, $\delta^{\widetilde{x}_0}_{\overline{y}_1}$ and $i^{\widetilde{y}_0}_{\widetilde{x}_0}$ when $x \to y$, subject to the relations

$$(5.4) \qquad \sum_{x \to y} r^{\overline{y}_1}_{\overline{x}_1} \delta^{\widetilde{x}_0}_{\overline{y}_1} = \sum_{z \to x} \delta^{\widetilde{z}_0}_{\overline{x}_1} i^{\widetilde{x}_0}_{\widetilde{z}_0}$$

for all $x \in X$.

Lemma 5.5. *In the category $\mathcal{ST}$, we have the relation*

$$\sum_{x \to y} r^{\overline{\{y\}}}_{\overline{\{x\}}} \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}} = \sum_{z \to x} \delta^{\widetilde{\{z\}}}_{\overline{\{x\}}} i^{\widetilde{\{x\}}}_{\widetilde{\{z\}}}$$

for all $x \in X$.

Proof. Since X is a UP space, the collections $\left(\overline{\{y\}}\right)_{x \to y}$ and $\left(\widetilde{\{z\}}\right)_{z \to x}$ are disjoint, respectively. Hence the desired relation simplifies to

$$r^{\overline{\partial}\{x\}}_{\overline{\{x\}}} \delta^{\widetilde{\{x\}}}_{\overline{\partial}\{x\}} = \delta^{\widetilde{\partial}\{x\}}_{\overline{\{x\}}} i^{\widetilde{\{x\}}}_{\widetilde{\partial}\{x\}},$$

which follows from Proposition 3.3(8) by setting $Y = \overline{\{x\}}$, $Z = \widetilde{\{x\}}$ and $W = \overline{\{x\}} \cup \widetilde{\{x\}}$. $\qquad\square$

Definition 5.6. The previous lemma allows us to define an additive functor $\mathcal{B} \to \mathcal{ST}$ by $\overline{x}_1 \mapsto (\overline{\{x\}}, 1)$ and $\widetilde{x}_0 \mapsto (\widetilde{\{x\}}, 0)$, and in the obvious way on morphisms. Let

$$\mathfrak{F}_{\mathcal{B}} \colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{B})$$

denote the induced functor. Define *filtered K-theory restricted to the canonical base*, $\mathrm{FK}_{\mathcal{B}} \colon \mathfrak{RR}(X) \to \mathfrak{Mod}(\mathcal{B})$, as the composition of $\mathrm{FK}_{\mathcal{ST}}$ with $\mathfrak{F}_{\mathcal{B}}$.

Remark 5.7. The invariant $\mathrm{FK}_{\mathcal{B}}$ is only defined for UP spaces as the boundary map $\delta^{\widetilde{\{x\}}}_{\overline{\{y\}}}$ only exists when $\overline{\{y\}} \cup \widetilde{\{x\}}$ belongs to $\mathbb{LC}(X)$. Also, the invariant $\mathrm{FK}_{\mathcal{B}}$ is most likely only sufficient for spaces where (5.14) holds for all boundary pairs (U, C).

Definition 5.8. A $\mathcal{B}$-module M is called *exact* if the sequence

$$(5.9) \qquad M(\overline{x}_1) \xrightarrow{\left(r^{\overline{y}_1}_{\overline{x}_1} \;\; -\delta^{\widetilde{z}_0}_{\overline{x}_1}\right)} \bigoplus_{x \to y} M(\overline{y}_1) \oplus \bigoplus_{z \to x} M(\widetilde{z}_0) \xrightarrow{\begin{pmatrix} \delta^{\widetilde{x}_0}_{\overline{y}_1} \\ i^{\widetilde{x}_0}_{\widetilde{z}_0} \end{pmatrix}} M(\widetilde{x}_0)$$

is exact for all $x \in X$.

Lemma 5.10. *If M is an exact $\mathcal{ST}$-module, then $\mathfrak{F}_{\mathcal{B}}(M)$ is an exact $\mathcal{B}$-module. In particular, if A is a C^*-algebra over X, then the $\mathcal{B}$-module $\mathrm{FK}_{\mathcal{B}}(A)$ is exact.*

Proof. Using again that the collections $\left(\overline{\{y\}}\right)_{x\to y}$ and $\left(\widetilde{\{z\}}\right)_{z\to x}$ are respectively disjoint, it suffices to prove exactness of the sequence

$$M(\overline{\{x\}},1) \xrightarrow{\left(r^{\overline{\partial}\{x\}}_{\overline{\{x\}}} \quad -\delta^{\widetilde{\partial}\{x\}}_{\overline{\{x\}}}\right)} M(\overline{\partial}\{x\},1) \oplus M(\widetilde{\partial}\{x\},0) \xrightarrow{\begin{pmatrix} \delta^{\widetilde{\{x\}}}_{\overline{\partial}\{x\}} \\ i^{\widetilde{\{x\}}}_{\widetilde{\partial}\{x\}} \end{pmatrix}} M(\widetilde{\{x\}},0),$$

which follows from a diagram chase through the commutative diagram

$$\begin{array}{ccccccc}
M(\{x\},1) & \longrightarrow & M(\overline{\{x\}},1) & \longrightarrow & M(\overline{\partial}\{x\},1) & \dashrightarrow & M(\{x\},0) \\
\| & & \downarrow & & \downarrow & & \| \\
M(\{x\},1) & \dashrightarrow & M(\widetilde{\partial}\{x\},0) & \longrightarrow & M(\widetilde{\{x\}},0) & \longrightarrow & M(\{x\},0)
\end{array}$$

whose rows are exact. $\qquad\square$

Definition 5.11. Let X be a finite T_0-space. A boundary pair (U,C) in X is called *elementary* if U and C are connected, U is open, C is closed and if, moreover, $U \subseteq \widetilde{C}$ and $C \subseteq \overline{U}$.

Definition 5.12. A UP space X is called *EBP* if every elementary boundary pair (U,C) in X is of the form $(\widetilde{\{x\}}, \overline{\{y\}})$ for two points x and y in X with $x \to y$.

Lemma 5.13. *Let X be an EBP space, and let (U,C) be a boundary pair in X. Then the following holds in $\mathcal{ST}$:*

$$(5.14) \qquad \delta^U_C = \sum_{x\to y, x\in U, y\in C} r^{\overline{\{y\}}\cap C}_C \; i^{\overline{\{y\}}}_{\overline{\{y\}}\cap C} \; \delta^{\widetilde{\{x\}}}_{\overline{\{y\}}} \; r^{\widetilde{\{x\}}\cap U}_{\widetilde{\{x\}}} \; i^U_{\widetilde{\{x\}}\cap U}.$$

Proof. We would like to show the relation (5.14) for a boundary pair (U,C) in X. The proof goes by the induction on the number $|U \cup C|$ of elements of $U \cup C$. If either U or C is empty, then both sides of (5.14) are 0. This takes care of the case $|U \cup C| = 0$. Suppose for a natural number n, we have shown (5.14) for all boundary pairs (U,C) with $|U \cup C| \le n$, and take a boundary pair (U,C) with $|U\cup C| = n+1$, arbitrarily. We are going to show (5.14) for this pair. If either U or C is empty, again both sides of (5.14) are zero. So we may assume that both U and C are non-empty. Suppose U is not connected, and choose two non-empty open and closed subsets U_1 and U_2 of U such that $U = U_1 \sqcup U_2$. Then for $i = 1,2$, (U_i,C) is a boundary pair with $|U_i \cup C| \le n$. Thus by the assumption of the induction, both (U_1,C) and (U_2,C) satisfy (5.14). Hence by (2), (7) and (3) of Proposition 3.3 we

have

$$\delta_C^U = \delta_C^U(r_U^{U_1} i_{U_1}^U + r_U^{U_2} i_{U_2}^U)$$

$$= \delta_C^{U_1} i_{U_1}^U + \delta_C^{U_2} i_{U_2}^U$$

$$= \Big(\sum_{x \to y, x \in U_1, y \in C} r_C^{\overline{\{y\}} \cap C} \; i_{\overline{\{y\}} \cap C}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U_1} \; i_{\widetilde{\{x\}} \cap U_1}^{U_1} \Big) i_{U_1}^U$$

$$+ \Big(\sum_{x \to y, x \in U_2, y \in C} r_C^{\overline{\{y\}} \cap C} \; i_{\overline{\{y\}} \cap C}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U_2} \; i_{\widetilde{\{x\}} \cap U_2}^{U_2} \Big) i_{U_2}^U$$

$$= \sum_{x \to y, x \in U, y \in C} r_C^{\overline{\{y\}} \cap C} \; i_{\overline{\{y\}} \cap C}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U} \; i_{\widetilde{\{x\}} \cap U}^{U}$$

since we for $x \in U_i$ have $\widetilde{\{x\}} \cap U_1 = \widetilde{\{x\}} \cap U$ because $U_i \subseteq U$ is open. This shows (5.14) for (U, C). Thus we may now assume U is connected. In a very similar way, we get (5.14) using the assumption of the induction if C is not connected. Thus we may assume C is connected. Next suppose we have $U \not\subseteq \widetilde{C}$. Set $U' = U \cap \widetilde{C}$ which is a proper open subset of U. The pair (U', C) is a boundary pair because $U' \cup C = (U \cup C) \cap \widetilde{C} \in \mathbb{LC}(X)$. We have $\delta_C^U = \delta_C^{U'} i_{U'}^U$ by applying (8) of Proposition 3.3 for $Y = C$, $Z = U$ and $W = U' \cup C$. Since $|U' \cup C| \le n$, we get by the assumption of the induction that

$$\delta_C^U = \delta_C^{U'} i_{U'}^U$$

$$= \Big(\sum_{x \to y, x \in U', y \in C} r_C^{\overline{\{y\}} \cap C} \; i_{\overline{\{y\}} \cap C}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U'} \; i_{\widetilde{\{x\}} \cap U'}^{U'} \Big) i_{U'}^U$$

$$= \sum_{x \to y, x \in U, y \in C} r_C^{\overline{\{y\}} \cap C} \; i_{\overline{\{y\}} \cap C}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U} \; i_{\widetilde{\{x\}} \cap U}^{U}$$

since $x \to y$, $x \in U$ and $y \in C$ imply $x \in U'$, and we have $\widetilde{\{x\}} \cap U = \widetilde{\{x\}} \cap U'$. This shows (5.14) for (U, C). Thus we may now assume $U \subseteq \widetilde{C}$. In a very similar way, we get (5.14) using the assumption of the induction if $C \not\subseteq \overline{U}$. Thus we may assume $C \subseteq \overline{U}$.

It remains to show (5.14) for a boundary pair (U, C) such that U and C are connected, $U \subseteq \widetilde{C}$ and $C \subseteq \overline{U}$. To this end, we use the assumption of the proposition. Take such a pair (U, C). Since X is UP, the pair $(\widetilde{U}, \overline{C})$ is a boundary pair by Lemma 5.1. It is not difficult to see that the pair $(\widetilde{U}, \overline{C})$ is elementary. Hence by the assumption of the proposition, there exist $x \in \widetilde{U}$ and $y \in \overline{C}$ such that $\widetilde{U} = \widetilde{\{x\}}$, $\overline{C} = \overline{\{y\}}$ and $x \to y$. By (6) and (7) of Proposition 3.3, we get

$$\delta_C^U = i_C^{\overline{C}} \delta_{\overline{C}}^{\widetilde{U}} r_{\widetilde{U}}^U = i_C^{\overline{\{y\}}} \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} r_{\widetilde{\{x\}}}^U.$$

It remains to prove that (x, y) is the only pair satisfying $x \to y$, $x \in U$ and $y \in C$. First note that $\widetilde{U} = \widetilde{\{x\}}$ implies $x \in U$, and also that $\overline{C} = \overline{\{y\}}$ implies $y \in C$. Now take $u \in U$ and $c \in C$ with $u \to c$. Since $U \subseteq \widetilde{\{x\}}$ and $C \subseteq \overline{\{y\}}$, there exist a path from u to x, and a path from y to c. These two paths together with the arrow $x \to y$ give us a path from u to c. Since X is UP, this path should coincide with the arrow $u \to c$. Hence we get $u = x$ and $c = y$. This finishes the proof. $\qquad \square$

Lemma 5.15. *Let X be a finite T_0-space. Assume that the directed graph associated to X is a* forrest, *i.e., it contains no undirected cycles. Then X is an EBP space.*

Proof. It is clear that, if the directed graph associated to X is a forrest, then X is a UP space. Let us take an elementary boundary pair (U, C). Choose a minimal element $x \in U$. Since $U \subseteq \widetilde{C}$, there is $y \in C$ with $x > y$. We can, moreover, assume that $x \to y$ because $U \cup C$ is locally closed and x is minimal in U. Since U is open and C is closed, we have $\widetilde{\{x\}} \subseteq U$ and $\overline{\{y\}} \subseteq C$. We will show that these inclusions are equalities using the fact that X is a forrest. Take $u \in U$ arbitrarily. Since $U \subseteq \widetilde{C}$, there exists a $c \in C$ such that $u > c$. Thus we have a path from u to c. Since both U and C are connected, there exist undirected paths from u to x and from y to c. These two paths give us an undirected path from u to c through the arrow $x \to y$. This path should coincide with the directed path from u to c because X contains no undirected cycles. Hence we get a path from u to x. This shows $u \in \widetilde{\{x\}}$, and therefore we get $U = \widetilde{\{x\}}$. In a similar manner, we get $C = \overline{\{y\}}$. $\square$

Remark 5.16. The above lemma applies, in particular, to accordion spaces. The conclusion of Lemma 5.13 can also be verified for various UP spaces which are not forrests—the smallest example being the so-called pseudocircle with four points. Consider, however, the sixteen-point space Q defined by the directed graph
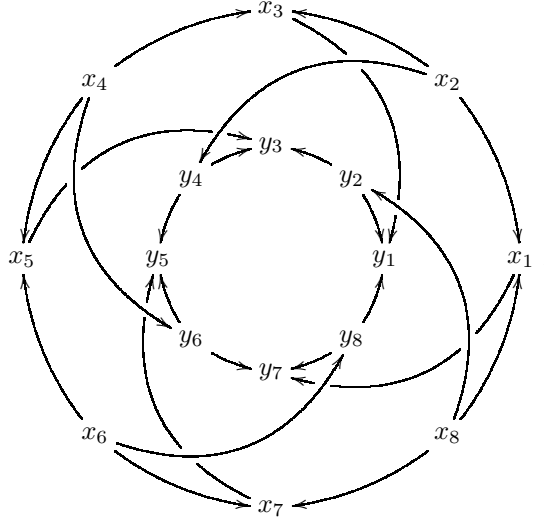
Then Q is a UP space that is not an EBP space as the subsets $U = \{x_1, x_2, \ldots, x_8\}$ and $C = \{y_1, y_2, \ldots, y_8\}$ give an elementary boundary pair (U, C) that does not satisfy $U = \widetilde{\{x\}}$ nor $C = \overline{\{y\}}$ for any $x, y \in X$. A simple computation shows that the boundary decomposition of δ_C^U specified in Lemma 5.13 holds in the category $\mathcal{NT}$. However, we believe that it does not hold in $\mathcal{ST}$.

The following theorem has two important consequences. Firstly, as stated in Corollary 5.19, it implies that for real rank zero C^*-algebras, isomorphisms on $\mathrm{FK}_{\mathcal{B}}$ lift to isomorphisms on $\mathrm{FK}_{\mathcal{ST}}$. By Theorem 3.11, $\mathrm{FK}_{\mathcal{ST}}$ is strongly complete for stable Kirchberg X-algebras when X is an accordion space. Secondly, by Lemma 5.6 of [5] any exact $\mathcal{NT}$-module over an accordion space X is of the form $\mathrm{FK}(A)$ for some Kirchberg X-algebra A, so any exact $\mathcal{B}$-module over the accordion space X

will be of the form $\mathrm{FK}_{\mathcal{B}}(A)$ for some Kirchberg X-algebra A of real rank zero. This second consequence is useful for constructing examples of Kirchberg X-algebras.

Theorem 5.17. *Let X be an EBP space. The functor*

$$\mathfrak{F}_{\mathcal{B}}\colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{B})$$

restricts to an equivalence between the category of exact real-rank-zero-like $\mathcal{ST}$-modules and the category of exact $\mathcal{B}$-modules.

A proof of this theorem is given after the following remark and corollary.

Remark 5.18. The proof works not only for EBP spaces but more generally for UP spaces for which

$$\delta_C^U = \sum_{x \to y, x \in U, y \in C} r_C^{\overline{\{y\}} \cap C}\ i_{\overline{\{y\}} \cap C}^{\widetilde{\{y\}}}\ \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}\ r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U}\ i_{\widetilde{\{x\}} \cap U}^{U}.$$

holds in $\mathcal{ST}$ for all boundary pairs (U, C), cf. Lemma 5.13.

Corollary 5.19. *Let A and B be C^*-algebras of real rank zero over an EBP space X. Then for any homomorphism $\varphi\colon \mathrm{FK}_{\mathcal{B}}(A) \to \mathrm{FK}_{\mathcal{B}}(B)$, there exists a unique homomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ for which $\mathfrak{F}_{\mathcal{B}}(\Phi) = \varphi$. If φ is an isomorphism, then so is Φ.*

Proof of Theorem 5.17. We will explicitly define a functor from the category of exact $\mathcal{B}$-modules to the category of exact real-rank-zero-like $\mathcal{ST}$-modules.

Let an exact $\mathcal{B}$-module N be given. We will define an $\mathcal{ST}$-module M. We begin in the obvious way: For $x \in X$, let $M(\overline{\{x\}}, 1) = N(\overline{x}_1)$ and $M(\widetilde{\{x\}}, 0) = N(\widetilde{x}_0)$. Similarly, for $x \to y$, we define the even component of $i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}}$ to be $i_{\widetilde{x}_0}^{\widetilde{y}_0}$, the odd component of $r_{\overline{\{x\}}}^{\overline{\{y\}}}$ to be $r_{\overline{x}_1}^{\overline{y}_1}$, and the odd-to-even component of $\delta_{\overline{\{y\}}}^{\widetilde{\{x\}}}$ to be $\delta_{\overline{y}_1}^{\widetilde{x}_0}$. This makes sure that, finally, we will have $\mathfrak{F}_{\mathcal{B}}(M) = N$. Also, we of course define $\delta_C^U\colon M(C, 0) \to M(U, 1)$ to be zero for every boundary pair (U, C) so that M will be real-rank-zero-like.

For $x \geq y$, let $x \to x_1 \to x_2 \to \cdots \to x_n \to y$ be the unique path from x to y. Define the even component of $i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}}$ to be the composition $i_{\widetilde{x}_0}^{\widetilde{x_1}_0} i_{\widetilde{x_1}_0}^{\widetilde{x_2}_0} \cdots i_{\widetilde{x_n}_0}^{\widetilde{y}_0}$ and the odd component of $r_{\overline{\{x\}}}^{\overline{\{y\}}}$ as the composition $r_{\overline{x}_1}^{\overline{x_1}_1} r_{\overline{x_1}_1}^{\overline{x_2}_1} \cdots r_{\overline{x_n}_1}^{\overline{y}_1}$. In case of $x = y$, this specifies to $i_{\widetilde{x}_0}^{\widetilde{x}_0} = \mathrm{id}_{M(\widetilde{\{x\}}, 0)}$ and $r_{\overline{x}_1}^{\overline{x}_1} = \mathrm{id}_{M(\overline{\{x\}}, 1)}$. If we have $x \to y$, then these definitions coincide with the ones we gave before.

We observe that the groups $M(\widetilde{\{x\}}, 0)$ with the maps $i_{\widetilde{\{x\}}}^{\widetilde{\{y\}}}$ define a precosheaf on $\mathbb{B} = \left\{ \widetilde{\{x\}} \mid x \in X \right\}$. By Lemma 4.8 it is in fact a cosheaf. We can therefore apply Lemma 4.7 and obtain groups $M(U, 0)$ for all sets U and maps $i_U^V\colon M(U, 0) \to M(V, 0)$ for open sets $U \subseteq V$ which fulfill the relations (1) and (3) in Proposition 3.3.

For an arbitrary locally closed subset $Y \in \mathbb{LC}(X)$ we write $Y = V \setminus U$ with open sets $U \subseteq V$ and define $M(Y, 0)$ as the cokernel of the map $i_U^V\colon M(U, 0) \to M(V, 0)$. That this definition does not depend on the choice of U and V can be seen in a way similar to the proof of [19, Lemma 2.15] using that pushouts of abelian groups preserve cokernels. We obtain maps $r_V^Y\colon M(V, 0) \to M(Y, 0)$ for every open set V

with relatively closed subset $Y \subseteq V$ such that the following holds: If $Y \in \mathbb{LC}(X)$ can be written as differences $V_i \setminus U_i$ of open sets for $i \in \{1, 2\}$ such that $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$, then the diagram

(5.20)
$$
\begin{array}{ccccc}
M(U_1, 0) & \xrightarrow{\ i\ } & M(V_1, 0) & \xrightarrow{\ r\ } & M(Y, 0) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \Vert \\
M(U_2, 0) & \xrightarrow{\ i\ } & M(V_2, 0) & \xrightarrow{\ r\ } & M(Y, 0)
\end{array}
$$

commutes.

For a relatively open subset $U \subseteq Y \in \mathbb{LC}(X)$ we obtain a map $i_U^Y \colon M(U, 0) \to M(Y, 0)$ using the diagram

(5.21)
$$
\begin{array}{ccccc}
M(\widetilde{\partial}U, 0) & \xrightarrow{\ i\ } & M(\widetilde{U}, 0) & \xrightarrow{\ r\ } & M(U, 0) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \vdots{\scriptstyle i} \\
M(\widetilde{\partial}Y, 0) & \xrightarrow{\ i\ } & M(\widetilde{Y}, 0) & \xrightarrow{\ r\ } & M(Y, 0).
\end{array}
$$

It is easy to check that this map coincides with the previously defined one in case Y is open.

We find that, for $Y_i \in \mathbb{LC}(X)$ with $Y_1 \subseteq Y_2$ open, and $Y_i = V_i \setminus U_i$ for $i \in \{1, 2\}$ and open sets U_i, V_i such that $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$, the diagram

(5.22)
$$
\begin{array}{ccccc}
M(U_1, 0) & \xrightarrow{\ i\ } & M(V_1, 0) & \xrightarrow{\ r\ } & M(Y_1, 0) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
M(U_2, 0) & \xrightarrow{\ i\ } & M(V_2, 0) & \xrightarrow{\ r\ } & M(Y_2, 0)
\end{array}
$$

commutes. We know this already for the left-hand square. For the right-hand square, it can be seen as follows: since V_1 is covered by U_1 and $\widetilde{Y_1}$, it suffices to check commutativity on the images $i_{U_1}^{V_1}(M(U_1, 0))$ and $i_{\widetilde{Y_1}}^{V_1}(M(\widetilde{Y_1}, 0))$. On $i_{U_1}^{V_1}(M(U_1, 0))$ both compositions vanish. On the image of $M(\widetilde{Y_1}, 0)$, commutativity follows from (5.20) and (5.21) considering the diagram

$$
\begin{array}{ccccc}
M(\widetilde{Y_1}, 0) & \xrightarrow{\ i\ } & M(V_1, 0) & \xrightarrow{\ r\ } & M(Y_1, 0) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
M(\widetilde{Y_2}, 0) & \xrightarrow{\ i\ } & M(V_2, 0) & \xrightarrow{\ r\ } & M(Y_2, 0).
\end{array}
$$

Now let $Y \in \mathbb{LC}(X)$, let U be a relatively open subset of Y and let $C = Y \setminus U$. Consider the diagram

(5.23)
$$
\begin{array}{ccccc}
M(\widetilde{\partial U}, 0) & \xrightarrow{\;i\;} & M(\widetilde{U}, 0) & \xrightarrow{\;r\;} & M(U, 0) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
M(\widetilde{\partial Y}, 0) & \xrightarrow{\;i\;} & M(\widetilde{Y}, 0) & \xrightarrow{\;r\;} & M(Y, 0) \\
\downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \vdots \\
M(\widetilde{\partial Y} \setminus \widetilde{\partial U}, 0) & \xrightarrow{\;i\;} & M(\widetilde{Y} \setminus \widetilde{U}, 0) & \xrightarrow{\;r\;} & M(C, 0),
\end{array}
$$

whose solid squares commute and whose rows and solid columns are exact. A diagram chase shows that there is a unique surjective map $r_Y^C \colon M(Y, 0) \to M(C, 0)$, as indicated by the dotted arrow, making the bottom-right square commute and making the right-hand column exact at $M(Y, 0)$. Again, we can easily check that this map coincides with the previously defined one in case Y is open.

We have now defined the even part of the module M completely. It is straightforward to check the relations (3) and (4) in Proposition 3.3. We will now prove that the relation (5) holds as well.

For this purpose, fix $Y \in \mathbb{LC}(X)$, let $U \subseteq Y$ be open and let $C \subseteq Y$ be closed. Consider the diagram

$$
\begin{array}{ccccc}
M(\widetilde{U}, 0) & \xrightarrow{\;r\;} & M(U, 0) & \xrightarrow{\;r\;} & M(U \cap C, 0) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
M(\widetilde{Y}, 0) & \xrightarrow{\;r\;} & M(Y, 0) & \xrightarrow{\;r\;} & M(C, 0)
\end{array}
$$

We would like to prove that the right-hand square commutes. The left-hand square commutes by definition of the map i_U^Y. Since $\widetilde{U} \cap C = U \cap C$, we can therefore assume without loss of generality that U and Y are open. Commutativity then follows from (5.22).

Next, we will convince ourselves that the relation (2) in Proposition 3.3 holds on the even part of M. Let $W = Y \sqcup Z$ be a topologically disjoint union of subsets $Y, Z \in \mathbb{LC}(X)$. Fix $w \in M(W, 0)$. Then $(w - wr_W^Z i_Z^W)r_W^Z = 0$ as $i_Z^W r_W^Z = \mathrm{id}_Z$. Hence there is $y \in M(Y, 0)$ with $y i_Y^W = w - wr_W^Z i_Z^W$. Applying r_W^Y shows $y = wr_W^Y$ as $i_Z^W r_W^Y = 0$. We get

$$
w(r_W^Y i_Y^W + r_W^Z i_Z^W) = y i_Y^W + wr_W^Z i_Z^W = w.
$$

We have shown that $r_W^Y i_Y^W + r_W^Z i_Z^W = \mathrm{id}_W$ as desired.

We have defined all even groups for the desired module M and the action of all transformations between them. We have checked all relations only involving transformations between even groups and verified exactness of $M(C, 0) \to M(Y, 0) \to M(U, 0)$ for every boundary pair $Y = U \cup C$.

We intend to do the same for the odd part of the module M in an analogous way. We start out with the given data consisting of the groups $M(\overline{\{x\}}, 1)$ and the maps $r_{\overline{x}_1}^{\overline{y}_1}$, $x \to y$, extend this to a sheaf on the basis $\{\overline{\{x\}} \mid x \in X\}$ of closed sets and apply Lemma 4.3. Observing that every locally closed subset of X can be written as a difference of two nested closed sets and using the functoriality of the kernel of

a group homomorphism, we define groups $M(\overline{Y}, 1)$ for all $Y \in \mathbb{LC}(X)$ and actions for all transformations between these odd groups. Using arguments analogous to the ones above, we can verify the relations (1) to (5) in Proposition 3.3 on the odd part of M.

It remains to define the odd-to-even components of the boundary maps δ_C^U for all boundary pairs (U, C), which has only been done in the special case $U = \widetilde{\{x\}}$, $C = \overline{\{y\}}$ with $x \to y$. Our general definition for $\delta_C^U \colon M(C, 1) \to M(U, 0)$ is

$$(5.24) \qquad \delta_C^U = \sum_{x \to y, x \in U, y \in C} r_C^{\overline{\{y\}} \cap C} \; i_{\overline{\{y\}} \cap C}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U} \; i_{\widetilde{\{x\}} \cap U}^{U}.$$

Our next aim is to verify the relations (6), (7) and (8) in Proposition 3.3. We begin with relation (6). Let (U, C) be a boundary pair and let $C' \subseteq C$ be relatively open. We have by the relations (3) and (5) that

$$i_{C'}^C \, \delta_C^U = i_{C'}^C \Big(\sum_{x \to y, x \in U, y \in C} r_C^{\overline{\{y\}} \cap C} \; i_{\overline{\{y\}} \cap C}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U} \; i_{\widetilde{\{x\}} \cap U}^{U} \Big)$$

$$= \sum_{x \to y, x \in U, y \in C} r_{C'}^{\overline{\{y\}} \cap C'} \; i_{\overline{\{y\}} \cap C'}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U} \; i_{\widetilde{\{x\}} \cap U}^{U}.$$

Since C' is relatively open in C, $\overline{\{y\}} \cap C'$ is empty unless $y \in C'$. Therefore, the above sum equals

$$\delta_{C'}^U = \sum_{x \to y, x \in U, y \in C'} r_{C'}^{\overline{\{y\}} \cap C'} \; i_{\overline{\{y\}} \cap C'}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap U} \; i_{\widetilde{\{x\}} \cap U}^{U}.$$

This shows relation (6). The relation (7) follows similarly.

Next we will check relation (8). Let $Y, Z, W \in \mathbb{LC}(X)$ such that $Y \cup W \in \mathbb{LC}(X)$ containing Y, W as closed subsets, $Z \cup W \in \mathbb{LC}(X)$ containing Z, W as open subsets, and $W \subseteq Y \cup Z$. For each $x \in Z$ and $y \in Y$ with $x \to y$, we define $\gamma_{x,y} \colon M(Y, 1) \to M(Z, 0)$ by

$$\gamma_{x,y} = r_Y^{\overline{\{y\}} \cap Y} \; i_{\overline{\{y\}} \cap Y}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap Z} \; i_{\widetilde{\{x\}} \cap Z}^{Z}.$$

Since $W \setminus Y$ is an open subset of Z (see the proof of Proposition 3.3), we have $\widetilde{\{x\}} \cap (W \setminus Y) = \widetilde{\{x\}} \cap Z$ for each $x \in W \setminus Y$. We also have $y \in W$ if $y \in Y$ satisfies $x \to y$ for some $x \in W \setminus Y$ because $W \subseteq Y \cup W$ is closed. Therefore, by the relation (4) we get

$$\delta_Y^{W \setminus Y} i_{W \setminus Y}^Z = \Big(\sum_{x \to y, x \in W \setminus Y, y \in Y} r_Y^{\overline{\{y\}} \cap Y} \; i_{\overline{\{y\}} \cap Y}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap (W \setminus Y)} \; i_{\widetilde{\{x\}} \cap (W \setminus Y)}^{W \setminus Y} \Big) i_{W \setminus Y}^Z$$

$$= \sum_{x \to y, x \in W \setminus Y, y \in W \cap Y} r_Y^{\overline{\{y\}} \cap Y} \; i_{\overline{\{y\}} \cap Y}^{\overline{\{y\}}} \; \delta_{\overline{\{y\}}}^{\widetilde{\{x\}}} \; r_{\widetilde{\{x\}}}^{\widetilde{\{x\}} \cap Z} \; i_{\widetilde{\{x\}} \cap Z}^{Z}$$

$$= \sum_{(x,y) \in \Lambda_1} \gamma_{x,y}$$

where we set

$$\Lambda_1 = \{(x, y) \mid x \to y, \; x \in W \setminus Y, \; y \in W \cap Y\}.$$

In a similar way using the facts that $W \setminus Z$ is a closed subset of Y and that Z is an open subset of $Y \cup Z$, we get

$$r_Y^{W \setminus Z} \delta_{W \setminus Z}^Z = \sum_{(x,y) \in \Lambda_2} \gamma_{x,y}$$

where we set

$$\Lambda_2 = \{(x,y) \mid x \to y, \ x \in W \cap Z, \ y \in W \setminus Z\}.$$

If we set

$$\Lambda_1' = \{(x,y) \mid x \to y, \ x \in W \cap Y \cap Z, \ y \in W \cap Y\},$$
$$\Lambda_2' = \{(x,y) \mid x \to y, \ x \in W \cap Z, \ y \in W \cap Y \cap Z\}$$

then we have

$$\{(x,y) \mid x \to y, \ x \in W \cap Z, \ y \in W \cap Y\} = \Lambda_1 \sqcup \Lambda_1' = \Lambda_2 \sqcup \Lambda_2'$$

because $W \subseteq Y \cup Z$ implies $(W \cap Z) \setminus Y = W \setminus Y$ and $(W \cap Y) \setminus Z = W \setminus Z$. Therefore in order to show the equality $\delta_Y^{W \setminus Y} i_{W \setminus Y}^Z = r_Y^{W \setminus Z} \delta_{W \setminus Z}^Z$, it suffices to show

$$\sum_{(x,y) \in \Lambda_1'} \gamma_{x,y} = \sum_{(x,y) \in \Lambda_2'} \gamma_{x,y}.$$

For each $p \in W \cap Y \cap Z$, we get

$$\sum_{y \leftarrow p} r_{\{p\}}^{\overline{\{y\}}} \delta_{\{y\}}^{\overline{\{p\}}} = \sum_{x \to p} \delta_{\{p\}}^{\overline{\{x\}}} i_{\{x\}}^{\overline{\{p\}}}$$

from the definition of $\mathcal{B}$-modules. Multiplying from the left with $r_Y^{\overline{\{p\}} \cap Y} i_{\{p\} \cap Y}^{\overline{\{p\}}}$ and from the right with $r_{\{p\}}^{\widetilde{\{p\}} \cap Z} i_{\widetilde{\{p\}} \cap Z}^Z$, and summing up over $p \in W \cap Y \cap Z$, we get

$$\sum_{p \in W \cap Y \cap Z} r_Y^{\overline{\{p\}} \cap Y} i_{\{p\} \cap Y}^{\overline{\{p\}}} \Big(\sum_{y \leftarrow p} r_{\{p\}}^{\overline{\{y\}}} \delta_{\{y\}}^{\overline{\{p\}}} \Big) r_{\{p\}}^{\widetilde{\{p\}} \cap Z} i_{\widetilde{\{p\}} \cap Z}^Z$$

$$= \sum_{p \in W \cap Y \cap Z} r_Y^{\overline{\{p\}} \cap Y} i_{\{p\} \cap Y}^{\overline{\{p\}}} \Big(\sum_{x \to p} \delta_{\{p\}}^{\overline{\{x\}}} i_{\{x\}}^{\overline{\{p\}}} \Big) r_{\{p\}}^{\widetilde{\{p\}} \cap Z} i_{\widetilde{\{p\}} \cap Z}^Z.$$

By the relations (3), (4) and (5), we get

$$(5.25) \qquad \sum_{p \in W \cap Y \cap Z} \sum_{y \leftarrow p} r_Y^{\overline{\{y\}} \cap Y} i_{\{y\} \cap Y}^{\overline{\{y\}}} \delta_{\{y\}}^{\overline{\{p\}}} r_{\{p\}}^{\widetilde{\{p\}} \cap Z} i_{\widetilde{\{p\}} \cap Z}^Z$$

$$= \sum_{p \in W \cap Y \cap Z} \sum_{x \to p} r_Y^{\overline{\{p\}} \cap Y} i_{\{p\} \cap Y}^{\overline{\{p\}}} \delta_{\{p\}}^{\overline{\{x\}}} r_{\{x\}}^{\widetilde{\{x\}} \cap Z} i_{\widetilde{\{x\}} \cap Z}^Z.$$

Since Y is locally closed, the conditions $p \in W \cap Y \cap Z$, $y \leftarrow p$ and $\overline{\{y\}} \cap Y \neq \emptyset$ imply $y \in Y$. This further implies $y \in W$ because $W \subseteq Y \cup W$ is closed. Hence the left-hand side of (5.25) equals $\sum_{(x,y) \in \Lambda_1'} \gamma_{x,y}$. In a similar way, we can see that the right-hand side of (5.25) equals $\sum_{(x,y) \in \Lambda_2'} \gamma_{x,y}$. Thus we have proven the relation (8), and this finishes the verification of all relations in Proposition 3.3.

Hence, M is indeed an $\mathcal{ST}$-module. To see that M is exact, it remains to show that the sequences $M(C,1) \xrightarrow{\delta_C^U} M(U,0) \xrightarrow{i_U^Y} M(Y,0)$ and $M(Y,1) \xrightarrow{r_Y^C} M(C,1) \xrightarrow{\delta_C^U} M(U,0)$ are exact for all boundary pairs (U,C) with $Y = U \cup C$.

Fix an element $x \in X$ and consider the commutative diagram

$$
\begin{array}{ccccc}
M(\{x\},1) & \xrightarrow{\ i\ } & M(\overline{\{x\}},1) & \xrightarrow{\ r\ } & M(\overline{\partial}\{x\},1) \\
\| & & \downarrow & & \downarrow \\
M(\{x\},1) & \dashrightarrow & M(\widetilde{\partial}\{x\},0) & \xrightarrow{\ i\ } & M(\widetilde{\{x\}},0)
\end{array}
$$

Using exactness of the upper row and the fact that N was an exact $\mathcal{B}$-module, a diagram chase shows that the bottom row is exact. In a similar way, we see that the sequence

$$
M(\overline{\{x\}},1) \to M(\overline{\partial}\{x\},0) \to M(\{x\},0).
$$

is exact for every $x \in X$.

Next, let $Y \in \mathbb{LC}(X)$ and let $x \in Y$ be a closed point. Then $Y \cap \widetilde{\{x\}}$ is relatively closed in $\widetilde{\{x\}}$ because Y is locally closed. A diagram chase in the commutative diagram

$$
\begin{array}{ccc}
M(\widetilde{\partial}\{x\} \setminus (Y \cap \widetilde{\partial}\{x\}),0) & =\!\!=\!\!= & M(\widetilde{\{x\}} \setminus (Y \cap \widetilde{\{x\}}),0) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
\end{array}
$$

$$
\begin{array}{ccccc}
M(\{x\},1) & \xrightarrow{\ \ } & M(\widetilde{\partial}\{x\},0) & \xrightarrow{\ i\ } & M(\widetilde{\{x\}},0) \\
\| & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\
M(\{x\},1) & \xrightarrow{\ \ } & M(Y \cap \widetilde{\partial}\{x\},0) & \xrightarrow{\ i\ } & M(Y \cap \widetilde{\{x\}},0),
\end{array}
$$

whose columns and top row are exact, yields exactness of the bottom row. By a diagram chase in the commutative diagram

$$
\begin{array}{ccccc}
M(\{x\},1) & \dashrightarrow & M(Y \cap \widetilde{\partial}\{x\},0) & \xrightarrow{\ i\ } & M(Y \cap \widetilde{\{x\}},0) \\
\| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
M(\{x\},1) & \dashrightarrow & M(Y \setminus \{x\},0) & \xrightarrow{\ i\ } & M(Y,0)
\end{array}
$$

using the exact cosheaf sequence (4.6) for the covering $(Y \setminus \{x\}, Y \cap \widetilde{\{x\}})$ of Y we obtain exactness of the bottom row. Notice that, using a further diagram chase, it is not hard to deduce the exactness of the cosheaf sequence for a relatively open covering of a locally closed set from the open case.

We have established the exactness of the sequence $M(C,1) \xrightarrow{\delta_C^U} M(U,0) \xrightarrow{i_U^Y} M(Y,0)$ for all boundary pairs (U,C) with C a singleton. Analogously, we find that $M(Y,1) \xrightarrow{r_Y^C} M(C,1) \xrightarrow{\delta_C^U} M(U,0)$ is exact whenever U is a singleton.

We will proceed by an inductive argument. Let $n \geq 1$ be a natural number and assume that exactness of the sequence $M(C,1) \xrightarrow{\delta_C^U} M(U,0) \xrightarrow{i_U^Y} M(Y,0)$ is proven for all boundary pairs (U,C) for which C has at most n elements. Let (U,C) be a boundary pair such that C has $n+1$ elements. Write $Y = U \cup C$. Let $p \in C$ be a maximal point and set $U' = U \cup \{p\}$, $C' = C \setminus \{p\}$. Then (U',C') is a boundary

pair. A diagram chase in the commutative diagram

$$
\begin{array}{ccccccc}
M(\{p\},1) & \xrightarrow{\ i\ } & M(C,1) & \xrightarrow{\ r\ } & M(C',1) & \multimap & M(\{p\},0) \\
\Big\| & & \Big\downarrow{\circ} & & \Big\downarrow{\circ} & & \Big\| \\
M(\{p\},1) & \multimap & M(U,0) & \xrightarrow{\ i\ } & M(U',0) & \xrightarrow{\ r\ } & M(\{p\},0) \\
& & \Big\downarrow{i} & & \Big\downarrow{i} & & \\
& & M(Y,0) & = & M(Y,0),
\end{array}
$$

whose rows and third column are exact, shows exactness of the second column. Again, exactness of $M(Y,1) \xrightarrow{r_Y^C} M(C,1) \xrightarrow{\delta_C^U} M(U,0)$ for all boundary pairs follows in a analogous manner. We conclude that M is an exact $\mathcal{ST}$-module.

Summing up, we have associated an exact real-rank-zero-like $\mathcal{ST}$-module with every exact $\mathcal{B}$-module. By the naturality of our constructions using kernels and cokernels we in fact obtain a functor G from the category of exact $\mathcal{B}$-modules to the category of exact real-rank-zero-like $\mathcal{ST}$-modules. Let F be the restriction of the functor $\mathfrak{F}_\mathcal{B}$ to the category of exact real-rank-zero-like $\mathcal{ST}$-modules. Then the composition GF is equal to the identity functor on the category of exact $\mathcal{B}$-modules. It remains to show that FG is naturally isomorphic to the identity functor on the category of exact real-rank-zero-like $\mathcal{ST}$-modules.

Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. We will construct a natural $\mathcal{ST}$-module isomorphism $\eta_M \colon M \to (FG)(M)$. For $x \in X$ we have $M(\widetilde{\{x\}},0) = (FG)(M)(\widetilde{\{x\}},0)$ and $M(\overline{\{x\}},1) = (FG)(M)(\overline{\{x\}},1)$. Hence we set $\eta_M(\widetilde{\{x\}},0) = \mathrm{id}_{M(\widetilde{\{x\}},0)}$ and $\eta_M(\overline{\{x\}},1) = \mathrm{id}_{M(\overline{\{x\}},1)}$. Using the universal property of kernels and cokernels we obtain natural group homomorphisms $f_Y \colon M(Y,1) \to (FG)(M)(Y,1)$ and $g_Y \colon (FG)(M)(Y,0) \to M(Y,0)$ for every $Y \in \mathbb{LC}(X)$. An application of the Five Lemma shows that these are in fact isomorphisms. We can therefore define $\eta_M(Y,1) = f_Y$ and $\eta_M(Y,0) = (g_Y)^{-1}$.

Finally, we check that this collection of maps constitutes an $\mathcal{ST}$-module homomorphism, that is, the group homomorphism $\eta_M \colon M \to (FG)(M)$ intertwines the actions of the category $\mathcal{ST}$ on M and on $(FG)(M)$. By construction this is true for the transformations $(i^{\widetilde{\{y\}}}_{\{x\}},0)$, $(r^{\overline{\{y\}}}_{\{x\}},1)$ and $\delta^{\widetilde{\{x\}}}_{\{y\}}$ for all $x,y \in X$ with $x \to y$. By Lemma 4.3 and Lemma 4.7 it is also true for the transformation $(i_U^V,0)$ for all open subset U,V of X with $U \subseteq V$ and for $(r_C^D,1)$ for all closed subsets C,D of X with $D \subseteq C$.

Let $V \subseteq X$ be open and let $Y \subseteq V$ be relatively closed. Since $(r_V^Y,0)$ was defined as a natural projection onto a cokernel, our assertion holds for this transformation as well. Consequently, by (5.21) the assertion also follows for the transformation $(i_U^Y,0)$ for $Y \in \mathbb{LC}(X)$ and $U \subseteq Y$ relatively open. Finally (5.23) implies the assertion for the transformation r_Y^C with $Y \in \mathbb{LC}(X)$ and $C \subseteq Y$ relatively closed. We have shown that η intertwines the actions of all even transformations on the 0-parts of M and $(FG)(M)$. By analogous arguments the same follows for the actions of all even transformations on the 1-parts of M and $(FG)(M)$.

Our last step is to consider the action of a boundary transformation δ_C^U for a boundary pair (U,C). Since M and $(FG)(M)$ are real-rank-zero-like the 0-to-1 component of δ_C^U acts trivially on both modules. We have already seen that the

assertion is true for the 1-to-0 component of δ_C^U in the specific case that $(U, C) = (\widetilde{\{x\}}, \widetilde{\{y\}})$ with $x \to y$. The general case then follows from (5.24) as X is EBP. $\square$

6. Reduced filtered K-theory

Let X be an arbitrary finite T_0-space. In this section we introduce a functor $\mathrm{FK}_{\mathcal{R}}$ which is equivalent to the reduced filtered K-theory defined by Gunnar Restorff in [22].

Definition 6.1. Let $\mathcal{R}$ denote the universal preadditive category generated by objects $x_1, \widetilde{\partial}x_0, \widetilde{x}_0$ for all $x \in X$ and morphisms $\delta_{x_1}^{\widetilde{\partial}x_0}$ and $i_{\widetilde{\partial}x_0}^{\widetilde{x}_0}$ for all $x \in X$, and $i_{\widetilde{y}_0}^{\widetilde{\partial}x_0}$ when $y \to x$, subject to the relations

$$\delta_{x_1}^{\widetilde{\partial}x_0} i_{\widetilde{\partial}x_0}^{\widetilde{x}_0} = 0 \tag{6.2}$$

$$i_p i_{\widetilde{y(p)}_0}^{\widetilde{\partial}x_0} = i_q i_{\widetilde{y(q)}_0}^{\widetilde{\partial}x_0} \tag{6.3}$$

for all $x \in X$, all $y \in X$ satisfying $y > x$, and all paths $p, q \in \mathrm{Path}(y, x)$, where for a path $p = (z_k)_{k=1}^n$ in $\mathrm{Path}(y, x)$, we define $y(p) = z_2$, and

$$i_p = i_{\widetilde{z}_{n_0}}^{\widetilde{\partial}z_{n-1_0}} i_{\widetilde{\partial}z_{n-1_0}}^{\widetilde{z_{n-2}}_0} \cdots i_{\widetilde{z}_{3_0}}^{\widetilde{\partial}z_{2_0}} i_{\widetilde{\partial}z_{2_0}}^{\widetilde{z}_{2_0}}.$$

Definition 6.4. It is easy to see that the relations in $\mathcal{ST}$ corresponding to (6.2) and (6.3) hold. We can thus define an additive functor $\mathcal{R} \to \mathcal{ST}$ by $x_1 \mapsto (\{x\}, 1)$, $\widetilde{\partial}x_0 \mapsto (\widetilde{\partial\{x\}}, 0)$ and $\widetilde{x}_0 \mapsto (\widetilde{\{x\}}, 0)$, and in the obvious way on morphisms. Let $\mathfrak{F}_{\mathcal{R}} \colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{R})$ denote the induced functor. Define *reduced filtered K-theory*, $\mathrm{FK}_{\mathcal{R}}$ as the composition of $\mathrm{FK}_{\mathcal{ST}}$ with $\mathfrak{F}_{\mathcal{R}}$.

Definition 6.5. An $\mathcal{R}$-module M is called *exact* if the sequences

$$M(x_1) \xrightarrow{\delta} M(\widetilde{\partial}x_0) \xrightarrow{i} M(\widetilde{x}_0) \tag{6.6}$$

$$\bigoplus_{(p,q) \in \mathrm{DP}(x)} M(\widetilde{z(p,q)}_0) \xrightarrow{(i_p - i_q)_{(p,q)}} \bigoplus_{y \to x} M(\widetilde{y}_0) \xrightarrow{(i_{\widetilde{y}_0}^{\widetilde{\partial}x_0})} M(\widetilde{\partial}x_0) \longrightarrow 0 \tag{6.7}$$

are exact for all $x \in X$, where $\mathrm{DP}(x)$ denotes the set of pairs of distinct paths (p, q) to x and from some common element which is denoted $z(p, q)$.

Lemma 6.8. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. Let Y be an open subset of X and let $(U_i)_{i \in I}$ be an open covering of Y. Then the following sequence is exact:*

$$\bigoplus_{i,j \in I} M(U_i \cap U_j, 0) \xrightarrow{(i_{U_i \cap U_j}^{U_i} - i_{U_i \cap U_j}^{U_j})} \bigoplus_{i \in I} M(U_i, 0) \xrightarrow{(i_{U_i}^Y)} M(Y, 0) \longrightarrow 0.$$

Proof. Using an inductive argument as in [9, Proposition 1.3], we can reduce to the case that I has only two elements. In this case, exactness follows from a straightforward diagram chase using the exact six-term sequences of the involved ideal inclusions analogous to the one in the proof of Lemma 5.10. $\square$

Corollary 6.9. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module and set $N = \mathfrak{F}_{\mathcal{R}}(M)$. Then N is an exact $\mathcal{R}$-module.*

Proof. We verify the exactness of the desired sequences in M. The sequence (6.6) is exact since it is part of the exact six-term sequence associated to the open inclusion $\widetilde{\partial\{x\}} \subseteq \widetilde{\{x\}}$. To prove exactness of the sequence (6.7), we apply the previous lemma to $Y = \widetilde{\partial}\{x\}$ and get the exact sequence

$$\bigoplus_{y\to x, y'\to x} M(\widetilde{\{y\}} \cap \widetilde{\{y'\}}, 0) \xrightarrow{\left(i^{\widetilde{\{y\}}}_{\widetilde{\{y\}}\cap\widetilde{\{y'\}}} - i^{\widetilde{\{y'\}}}_{\widetilde{\{y\}}\cap\widetilde{\{y'\}}} \right)} \bigoplus_{y\to x} M(\widetilde{\{y\}}, 0)$$

$$\xrightarrow{\left(i^{\widetilde{\partial}\{x\}}_{\widetilde{\{y\}}} \right)} M(\widetilde{\partial}\{x\}, 0) \longrightarrow 0.$$

Another application of the previous lemma shows that $\bigoplus_{(p,q)\in\mathrm{DP}(x)} M(\widetilde{z(p,q)}, 0)$ surjects onto $\bigoplus_{y\to x, y'\to x} M(\widetilde{\{y\}} \cap \widetilde{\{y'\}}, 0)$ in a way making the obvious triangle commute. This establishes the exact sequence (6.7). $\square$

Remark 6.10. If X is a UP space, then the set $\mathrm{DP}(x)$ is empty for every $x \in X$. Hence, for an exact $\mathcal{R}$-module M, the map $(i^{\widetilde{\partial}x_0}_{\widetilde{y}_0})\colon \bigoplus_{y\to x} M(\widetilde{y}_0) \to M(\widetilde{\partial}x_0)$ is an isomorphism. In this sense, the groups $M(\widetilde{\partial}x_0)$ are redundant for an exact $\mathcal{R}$-module in case X is UP.

7. An intermediate invariant

In this section, we define one more invariant, which, in a sense, can be thought of as a union or join of reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ and filtered K-theory restricted to canonical base $\mathrm{FK}_{\mathcal{B}}$. It functions as an intermediate invariant towards concrete filtered K-theory $\mathrm{FK}_{\mathcal{ST}}$.

Let X be a UP space.

Definition 7.1. Let $\mathcal{BR}$ denote the universal preadditive category generated by objects x_1, $\overline{x}_1$, $\widetilde{x}_0$ for all $x \in X$ and morphisms $i^{\overline{x}_1}_{x_1}$ for all $x \in X$ and $r^{\overline{y}_1}_{\overline{x}_1}$, $\delta^{\widetilde{x}_0}_{\overline{y}_1}$ and $i^{\widetilde{y}_0}_{\widetilde{x}_0}$ when $x \to y$, subject to the relations

$$(7.2) \qquad\qquad \sum_{x\to y} r^{\overline{y}_1}_{\overline{x}_1} \delta^{\widetilde{x}_0}_{\overline{y}_1} = \sum_{z\to x} \delta^{\widetilde{z}_0}_{\overline{x}_1} i^{\widetilde{x}_0}_{\widetilde{z}_0}$$

for all $x \in X$ and

$$(7.3) \qquad\qquad i^{\overline{x}_1}_{x_1} r^{\overline{y}_1}_{\overline{x}_1} = 0$$

when $x \to y$.

As before, there is a canonical additive functor $\mathcal{BR} \to \mathcal{ST}$, inducing a functor $\mathfrak{F}_{\mathcal{BR}}\colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{BR})$. Define $\mathrm{FK}_{\mathcal{BR}}$ as the composition of $\mathrm{FK}_{\mathcal{ST}}$ with $\mathfrak{F}_{\mathcal{BR}}$.

The category $\mathcal{B}$ embeds into $\mathcal{BR}$, and a forgetful functor $\mathfrak{Mod}(\mathcal{BR}) \to \mathfrak{Mod}(\mathcal{B})$ is induced. Define an additive functor $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}\colon \mathfrak{Mod}(\mathcal{BR}) \to \mathfrak{Mod}(\mathcal{R})$ by

$$M(\widetilde{\partial}x_0) = \bigoplus_{y\to x} M(\widetilde{y}_0)$$

and $\delta^{\widetilde{\partial}x_0}_{x_1} = (i^{\overline{x}_1}_{x_1} \delta^{\widetilde{y}_0}_{\overline{x}_1})$. One can check that this functor is well-defined.

Definition 7.4. A $\mathcal{BR}$-module M is called *exact* if the sequences

$$(7.5) \qquad M(\overline{x}_1) \xrightarrow{\left(r_{\overline{x}_1}^{\overline{y}_1} \quad -\delta_{\overline{x}_1}^{\widetilde{z}_0}\right)} \bigoplus_{x \to y} M(\overline{y}_1) \oplus \bigoplus_{z \to x} M(\widetilde{z}_0) \xrightarrow{\begin{pmatrix} \delta_{\overline{y}_1}^{\widetilde{x}_0} \\ i_{\widetilde{z}_0}^{\widetilde{x}_0} \end{pmatrix}} M(\widetilde{x}_0)$$

$$(7.6) \qquad 0 \to M(x_1) \xrightarrow{i_{x_1}^{\overline{x}_1}} M(\overline{x}_1) \xrightarrow{\left(r_{\overline{x}_1}^{\overline{y}_1}\right)} \bigoplus_{x \to y} M(\overline{y}_1)$$

are exact for all $x \in X$ and all $y \in X$ satisfying $x \to y$.

Lemma 7.7. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. Then $\mathfrak{F}_{\mathcal{BR}}(M)$ is an exact $\mathcal{BR}$-module.*

Proof. The proof is similar to the proof of Lemma 5.10. $\qquad\square$

Theorem 7.8. *Assume that X is a UP space. Let M and N be exact $\mathcal{BR}$-modules with $M(x_1)$ and $N(x_1)$ free for all non-open points $x \in X$, and let $\varphi \colon \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(M) \to \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(N)$ be an $\mathcal{R}$-module homomorphism. Then there exists a (not necessarily unique) $\mathcal{BR}$-module homomorphism $\Phi \colon M \to N$ satisfying $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}(\Phi) = \varphi$. If φ is an isomorphism then, by construction, so is Φ.*

Proof. For $x \in X$, we define $\Phi_{x_1} = \varphi_{x_1}$ and $\Phi_{\widetilde{x}_0} = \varphi_{\widetilde{x}_0}$. In the following, we will define $\Phi_{\overline{x}_1}$ by induction on the partial order of X in a way such that the relations

$$(7.9) \qquad\qquad r_{\overline{x}_1}^{\overline{y}_1} \Phi_{\overline{y}_1} = \Phi_{\overline{x}_1} r_{\overline{x}_1}^{\overline{y}_1},$$

$$(7.10) \qquad\qquad \delta_{\overline{x}_1}^{\widetilde{z}_0} \Phi_{\widetilde{z}_0} = \Phi_{\overline{x}_1} \delta_{\overline{x}_1}^{\widetilde{z}_0}$$

$$(7.11) \qquad\qquad i_{x_1}^{\overline{x}_1} \Phi_{\overline{x}_1} = \Phi_{x_1} i_{x_1}^{\overline{x}_1}$$

hold for all y with $x \to y$ and all z with $z \to x$. For closed points $x \in X$, we set

$$\Phi_{\overline{x}_1} = i_{x_1}^{\overline{x}_1} \varphi_{x_1} \left(i_{x_1}^{\overline{x}_1}\right)^{-1}.$$

Here we have used that, by exactness of (7.6), $i_{x_1}^{\overline{x}_1}$ is invertible as there is no y with $x \to y$. While the condition (7.9) is empty, (7.10) is guaranteed by φ being an $\mathcal{R}$-module homomorphism, and (7.11) holds by construction.

Now fix an element $w \in X$ and assume that $\Phi_{\overline{x}_1}$ is defined for all $x < w$ in a way such that (7.9) and (7.10) hold. Using the exact sequence (7.6) and the freeness of $\bigoplus_{w \to x} M(\overline{w}_1)$, we can choose a free subgroup $V \subseteq M(\overline{w}_1)$ such that $M(\overline{w}_1)$ decomposes as an inner direct sum

$$M(\overline{w}_1) = V \oplus M(w_1) \cdot i_{w_1}^{\overline{w}_1}.$$

We will define $\Phi_{\overline{w}_1}$ by specifying the two restrictions $\Phi_{\overline{w}_1}|_V$ and $\Phi_{\overline{w}_1}|_{M(w_1)\cdot i_{w_1}^{\overline{w}_1}}$. Consider the diagram

(7.12)

$$
\begin{array}{ccccc}
V \rightarrowtail & M(\overline{x}_1) \xrightarrow{\left(r_{\overline{x}_1}^{\overline{y}_1}, -\delta_{\overline{x}_1}^{\widetilde{z}_0}\right)} \bigoplus_{x\to y} M(\overline{y}_1) \oplus \bigoplus_{z\to x} M(\widetilde{z}_0) & \xrightarrow{\binom{\delta_{\overline{y}_1}^{\widetilde{x}_0}}{i_{\widetilde{z}_0}^{\widetilde{x}_0}}} & M(\widetilde{x}_0) \\
& \downarrow{\scriptstyle ((\Phi_{\overline{y}_1}),(\Phi_{\widetilde{z}_0}))} & & \downarrow{\scriptstyle \Phi_{\widetilde{x}_0}} \\
N(\overline{x}_1) \xrightarrow[\left(r_{\overline{x}_1}^{\overline{y}_1}, -\delta_{\overline{x}_1}^{\widetilde{z}_0}\right)]{} & \bigoplus_{x\to y} N(\overline{y}_1) \oplus \bigoplus_{z\to x} N(\widetilde{z}_0) & \xrightarrow[\binom{\delta_{\overline{y}_1}^{\widetilde{x}_0}}{i_{\widetilde{z}_0}^{\widetilde{x}_0}}]{} & N(\widetilde{x}_0)
\end{array}
$$

By assumption, the rows of this diagram are exact and the right-hand square commutes. We can therefore choose a homomorphism $\Phi_{\overline{x}_1}|_V\colon V \to N(\overline{x}_1)$ such that the left-hand pentagon commutes.

By exactness of (7.6), $i_{x_1}^{\overline{x}_1}$ is injective. Its corestriction onto its image $M(x_1)\cdot i_{x_1}^{\overline{x}_1}$ is thus an isomorphism. We may therefore define the restriction $\Phi_{\overline{x}_1}|_{M(x_1)\cdot i_{x_1}^{\overline{x}_1}}$ in the unique way that makes the following diagram commute:

(7.13)

$$
\begin{array}{ccc}
M(x_1) & \xrightarrow{i_{x_1}^{\overline{x}_1}} & M(x_1)\cdot i_{x_1}^{\overline{x}_1} \\
\downarrow{\scriptstyle \varphi_{x_1}} & & \downarrow{\scriptstyle \Phi_{\overline{x}_1}|_{M(x_1)\cdot i_{x_1}^{\overline{x}_1}}} \\
N(x_1) & \xrightarrow{i_{x_1}^{\overline{x}_1}} & N(x_1)\cdot i_{x_1}^{\overline{x}_1}
\end{array}
$$

We have to check that $\Phi_{\overline{w}_1} = (\Phi_{\overline{w}_1}|_V, \Phi_{\overline{w}_1}|_{M(w_1)\cdot i_{w_1}^{\overline{w}_1}})$ fulfills (7.9) and (7.10) (with x replaced with w). This is true on V because of the commutativity of the left-hand side of (7.12). It is also true on the second summand: by (7.3), both sides of (7.9) vanish on this subgroup; (7.10) follows again from φ being an $\mathcal{R}$-module homomorphism; and (7.11) holds by construction. This completes the induction step.

The claim, that Φ is an isomorphism whenever φ is, follows from a repeated application of the Five Lemma. $\square$

Corollary 7.14. *Assume that X is an EBP space. Let M and N be exact, real-rank-zero-like $\mathcal{ST}$-modules with $M(\{x\},1)$ and $N(\{x\},1)$ free for all non-open points $x \in X$, and let $\varphi\colon \mathfrak{F}_{\mathcal{R}}(M) \to \mathfrak{F}_{\mathcal{R}}(N)$ be an $\mathcal{R}$-module homomorphism. Then there exists a (not necessarily unique) $\mathcal{ST}$-module homomorphism $\Phi\colon M \to N$ satisfying $\mathfrak{F}_{\mathcal{R}}(\Phi) = \varphi$, and if Φ is by construction an isomorphism.*

*Proof.* Combine Theorems 7.8 and 5.17. $\square$

Corollary 7.15. *Let A and B be C^*-algebras of real rank zero over an EBP space X, and assume that $\mathrm{K}_1(A(x))$ and $\mathrm{K}_1(B(x))$ are free groups for all non-open points $x \in X$. Then for any homomorphism $\varphi\colon \mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$, there exist a (not necessarily unique) homomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ for which $\mathfrak{F}_{\mathcal{R}}(\Phi) = \varphi$. If φ is an isomorphism, then Φ is by construction an isomorphism.*

Consider the space $\mathcal{D} = \{1,2,3,4\}$ defined by $4 \to 3, 4 \to 2, 3 \to 1, 2 \to 1$. The space $\mathcal{D}$ is not a UP space. The second named author showed in [4] that there exists a finite refinement FK' of filtered K-theory FK given by adding a C^*-algebra $R_{1\backslash 4}$

to the collection $(R_Y)_{Y\in\mathbb{LC}(\mathcal{D})^*}$ of representing objects, creating a larger category $\mathcal{NT}'$. By [4, Theorem 6.2.14], isomorphisms on the refined filtered K-theory FK' lift to $\mathrm{KK}(\mathcal{D})$-equivalences, and thereby (using [16]) to $\mathcal{D}$-equivariant isomorphisms, for stable Kirchberg $\mathcal{D}$-algebras with all simple subquotients in the bootstrap class. However, there exists non-isomorphic stable Kirchberg $\mathcal{D}$-algebras A and B of real rank zero and with $\mathrm{FK}(A) \cong \mathrm{FK}(B)$, cf. [2].

Proposition 7.16. *Let A and B be C^*-algebras over $\mathcal{D}$, assume that A and B have real rank zero, and assume that $\mathrm{K}_1(A(x))$ and $\mathrm{K}_1(B(x))$ are free groups for all $x \in \{1,2,3\}$. Then any homomorphism $\varphi\colon \mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ extends (non-uniquely) to a homomorphism $\Phi\colon \mathrm{FK}'(A) \to \mathrm{FK}'(B)$. If φ is an isomorphism, then Φ is by construction an isomorphism.*

Proof. By Section 6.2.5 of [4], the refined filtered K-theory FK' consists of the following groups and maps:
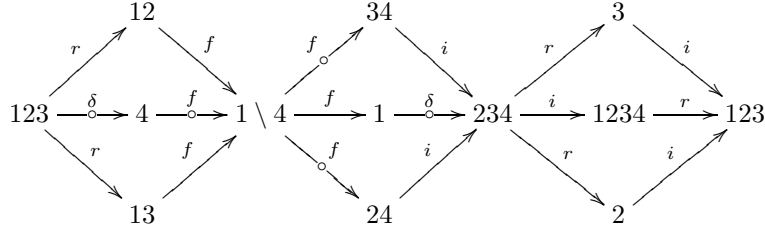
The proof of [2, Lemma 3.4] applies to the space $\mathcal{D}$, hence the two triangles

$$\mathrm{FK}_{234}(A) \xrightarrow{r^3_{234} i^{123}_3 \delta^4_{123} f_4} \mathrm{FK}_{1\backslash 4}(A)$$

with maps $i^{234}_{34}+\delta^{234}_1+i^{234}_{24}$ and (f^{34},f^1,f^{24}) to

$$\mathrm{FK}_{34}(A) \oplus \mathrm{FK}_1(A)[1] \oplus \mathrm{FK}_{23}(A)$$

$$\mathrm{FK}_{1\backslash 4}(A) \xrightarrow{f^1 \delta^{234}_1 r^3_{234} i^{123}_3} \mathrm{FK}_{123}(A)$$

with maps $f_{12}+f_4+f_{13}$ and $(r^{12}_{123},\delta^4_{123},r^{13}_{123})$ to

$$\mathrm{FK}_{12}(A) \oplus \mathrm{FK}_4(A)[1] \oplus \mathrm{FK}_{13}(A)$$

are exact. Since A is of real rank zero, the maps

$$\mathrm{FK}^0_{123}(A) \xrightarrow{\delta^4_{123}} \mathrm{FK}^1_4(A), \quad \mathrm{FK}^0_{12}(A) \xrightarrow{\delta^{34}_{12}} \mathrm{FK}^1_{34}(A),$$

$$\mathrm{FK}^0_{13}(A) \xrightarrow{\delta^{24}_{13}} \mathrm{FK}^1_{24}(A), \quad \mathrm{FK}^0_1(A) \xrightarrow{\delta^{234}_1} \mathrm{FK}^1_{234}(A)$$

vanish by Proposition 4 of [17]. So for C^*-algebras over $\mathcal{D}$ of real rank zero, the invariant FK' with the group $1 \backslash 4_0$ and its related maps omitted, consists of the

following groups and maps:
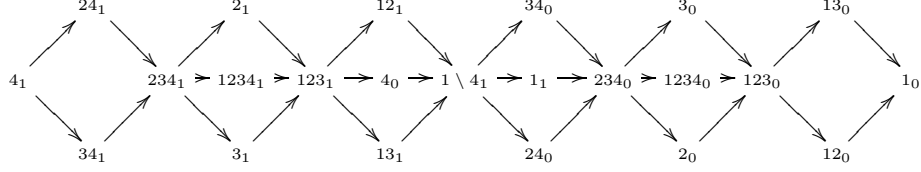
The reduced filtered K-theory $FK_{\mathcal{R}}$ consists of the sequences $3_1 \to 4_0 \to 34_0$, $2_1 \to 4_0 \to 24_0$, $1_1 \to 234_0 \to 1234_0$ together with the maps $34_0 \to 234_0$ and $24_0 \to 234_0$ and the group 4_1.

We will now construct $\Phi = (\Phi_Y^*)_{Y \in \mathbb{LC}(\mathcal{D})^* \cup \{1\backslash 4\}}$ from φ. Define $\Phi_{\{x\}}^1 = \varphi_{x_1}$, $\Phi_{\partial x}^0 = \varphi_{\widetilde{\partial x_0}}$, and $\Phi_{\widetilde{\{x\}}}^0 = \varphi_{\widetilde{x_0}}$ for all $x \in \mathcal{D}$. For $Y \in \{3, 2, 123, 13, 12, 1\}$, the maps Φ_Y^0 are constructed as the induced maps on cokernels, as in the proof of Theorem 5.17.

Since $FK_1^1(A)$ is free and the sequence

$$0 \longrightarrow FK_4^0(A) \xrightarrow{f_4} FK_{1\backslash 4}^1(A) \xrightarrow{f^1} FK_1^1(A) \longrightarrow 0$$

is exact, we can find a free subgroup $V_{1\backslash 4}$ of $FK_{1\backslash 4}^1(A)$ for which $\operatorname{im} f_4 \oplus V_{1\backslash 4} = FK_{1\backslash 4}^1(A)$. Consider the commuting diagram

$$
\begin{array}{ccccc}
V_{1\backslash 4} & \xrightarrow{(f^{34}, f^1, f^{24})} & FK_{34}^0(A) \oplus FK_1^1(A) \oplus FK_{24}^0(A) & \xrightarrow{i_{34}^{234} + \delta_1^{234} + i_{24}^{234}} & FK_{234}^0(A) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi_{34}^0 \oplus \Phi_1^1 \oplus \Phi_{24}^0} & & \Big\downarrow{\scriptstyle \Phi_{234}^0} \\
FK_{1\backslash 4}^1(B) & \xrightarrow{(f^{34}, f^1, f^{24})} & FK_{34}^0(B) \oplus FK_1^1(B) \oplus FK_{24}^0(B) & \xrightarrow{i_{34}^{234} + \delta_1^{234} + i_{24}^{234}} & FK_{234}^0(B).
\end{array}
$$

Since the bottom row is exact and due to freeness of $V_{1\backslash 4}$, we may choose a map $\psi \colon V_{1\backslash 4} \to FK_{1\backslash 4}^1(B)$ that makes the left square of the diagram commute. Define $\Phi_{1\backslash 4}^1$ on $\operatorname{im} f_4 \oplus V_{1\backslash 4}$ as $\Phi_4^0 + \psi$. By construction,

$$\Phi_{1\backslash 4}^1 f_4 = f_4 \Phi_4^0, \quad f^1 \Phi_{1\backslash 4}^1 = \Phi_1^1 f^1, \quad f^{34} \Phi_{1\backslash 4}^1 = \Phi_{34}^0 f^{34}, \quad f^{24} \Phi_{1\backslash 4}^1 = \Phi_{24}^0 f^{24},$$

and by the Five Lemma, the homomorphism $\Phi_{1\backslash 4}^1$ is an isomorphism if φ is an isomorphism.

Similarly, to construct Φ_{12}^1, use exactness of the sequence

$$0 \longrightarrow FK_2^1(A) \xrightarrow{i_2^{12}} FK_{12}(A) \xrightarrow{r_{12}^1} FK_1^1(A) \xrightarrow{\delta_1^2} FK_2^0(A)$$

and freeness of $FK_1^1(A)$ to choose a free subgroup V_{12} of $FK_{12}^1(A)$ for which $\operatorname{im} i_2^{12} \oplus V_{12} = FK_{12}^1(A)$. Using exactness of the bottom row of the commuting diagram

$$
\begin{array}{ccccc}
V_{12} & \xrightarrow{f_{12}} & FK_{1\backslash 4}^1(A) & \xrightarrow{f^{24}} & FK_{24}^0(A) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi_{1\backslash 4}^1} & & \Big\downarrow{\scriptstyle \Phi_{24}^0} \\
FK_{12}^1(B) & \xrightarrow{f_{12}} & FK_{1\backslash 4}^1(B) & \xrightarrow{f^{24}} & FK_{24}^0(B)
\end{array}
$$

the map Φ_{12}^1 can be constructed so that

$$\Phi_{12}^1 i_2^{12} = i_2^{12} \Phi_2^1, \quad f_{12} \Phi_{12}^1 = \Phi_{1\backslash 4}^1 f_{12}.$$

Again due to the Five Lemma, Φ_{12}^1 is an isomorphism if φ is. The maps Φ_{13}^1, Φ_{123}^1, Φ_{1234}^1, Φ_{234}^1, Φ_{34}^1, and Φ_{24}^1 are constructed similarly and in the specified order.

Finally, the group $\mathrm{FK}_{1\backslash 4}^0(A)$ is naturally isomorphic to

$$\mathrm{coker}(\mathrm{FK}_{123}^0(A) \xrightarrow{(r_{123}^{12}, \delta_{123}^4, r_{123}^{13})} \mathrm{FK}_{12}^0(A) \oplus \mathrm{FK}_4^1(A) \oplus \mathrm{FK}_{13}^1(A))$$

$$= \mathrm{FK}_4^1(A) \oplus \mathrm{coker}(\mathrm{FK}_{123}^0(A) \xrightarrow{(r_{123}^{12}, r_{123}^{13})} \mathrm{FK}_{12}^0(A) \oplus \mathrm{FK}_{13}^1(A))$$

whose second summand, due to real rank zero, is naturally isomorphic to $\mathrm{FK}_1^0(A)$ Therefore, by defining $\Phi_{1\backslash 4}^0$ as the map induced by $\Phi_4^1 \oplus \Phi_1^0$, Φ becomes a $\mathcal{NT}'$-morphism. $\qquad\square$

Corollary 7.17. *Let A and B be C^*-algebras over $\mathcal{D}$. Assume that A and B have real rank zero, that $\mathrm{K}_1(A(x))$ and $\mathrm{K}_1(B(x))$ are free groups for all $x \in \{1, 2, 3\}$, and that A and B are in the bootstrap class of Meyer-Nest. Then any isomorphism $\mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ lifts to a $\mathrm{KK}(\mathcal{D})$-equivalence.*

Proof. Combine Proposition 7.16 with [4, Theorem 6.2.14]. $\qquad\square$

Corollary 7.18. *Let A and B be stable Kirchberg $\mathcal{D}$-algebras of real rank zero, assume that $\mathrm{K}_1(A(x))$ and $\mathrm{K}_1(B(x))$ are free groups for all $x \in \{1, 2, 3\}$, and assume that $A(x)$ and $B(x)$ are in the bootstrap class for all $x \in \mathcal{D}$. Then any isomorphism $\mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ lifts to an isomorphism $A \to B$.*

Proof. Combine Proposition 7.16 with [4, Theorem 6.2.15]. $\qquad\square$

8. RANGE OF REDUCED FILTERED K-THEORY

Let X be an arbitrary finite T_0-space. In this section, the range of reduced filtered K-theory for purely infinite graph algebras is determined.

Let E be a countable graph and assume that all vertices in E are regular and support at least two cycles. Recall that a cycle is an edge whose source and range coincide. Recall also that the saturated, hereditary subsets of E^0 correspond to ideals in $C^*(E)$. Since all subsets of E^0 are saturated, a continuous map $\mathrm{Prim}(C^*(E)) \to X$ corresponds to a map $\psi: E^0 \to X$ satisfying $\psi(s(e)) \geq \psi(r(e))$ for all $e \in E^1$.

Assume that such a map ψ is given, i.e., that $C^*(E)$ is a C^*-algebra over X. Then $\mathrm{FK}_{\mathcal{R}}(C^*(E))$ can be computed in the following way. Define for each $F \subseteq X$ a matrix $D_F \in M_{\psi^{-1}(F)}(\mathbb{Z}_+)$ as $D_F = A_F - 1$ where A_F is defined as

$$A_F(v, w) = |\{e \in E^1 \mid r(e) = v, s(e) = w\}|.$$

Let $Y \in \mathbb{LC}(X)$ and $U \in \mathbb{O}(Y)$ be given, and define $C = Y \setminus U$. Then by [11], the six-term exact sequence induced by $C^*(E)(U) \hookrightarrow C^*(E)(Y) \twoheadrightarrow C^*(E)(C)$ is naturally isomorphic to the sequence

$$
\begin{array}{ccccc}
\mathrm{coker}\, D_U & \longrightarrow & \mathrm{coker}\, D_Y & \longrightarrow & \mathrm{coker}\, D_C \\
{\scriptstyle D_Y|_{\psi^{-1}(C)}^{\psi^{-1}(U)}}\big\uparrow & & & & \big\downarrow{\scriptstyle 0} \\
\ker D_C & \longleftarrow & \ker D_Y & \longleftarrow & \ker D_U
\end{array}
$$

induced, via the Snake Lemma, by the commuting diagram

$$
\begin{array}{ccccc}
\mathbb{Z}^{\psi^{-1}(U)} & \longrightarrow & \mathbb{Z}^{\psi^{-1}(Y)} & \longrightarrow & \mathbb{Z}^{\psi^{-1}(C)} \\
\big\downarrow {\scriptstyle D_U} & & \big\downarrow {\scriptstyle D_Y} & & \big\downarrow {\scriptstyle D_C} \\
\mathbb{Z}^{\psi^{-1}(U)} & \longrightarrow & \mathbb{Z}^{\psi^{-1}(Y)} & \longrightarrow & \mathbb{Z}^{\psi^{-1}(C)}.
\end{array}
$$

Given a map $\psi\colon E^0 \to X$, one can define matrices D_F as above. Then $C^*(E)$ is a C^*-algebra over X, via ψ, if and only if $D_X|_{\psi^{-1}(y)}^{\psi^{-1}(z)}$ vanishes when $y \not< z$. And if furthermore $D_X|_{\psi^{-1}(y)}^{\psi^{-1}(z)}$ is non-zero whenever $y < z$, then $C^*(E)$ is tight over X.

The following theorem by Søren Eilers, Mark Tomforde, James West and the third named author, determines the range of filtered K-theory over the two-point space $\{1, 2\}$ with $2 \to 1$. We quote it here to apply it in the proof of Theorem 8.2.

Theorem 8.1 ([13, Propositions 4.3 and 4.7]). *Let $\mathcal{E}$*

$$
\begin{array}{ccccc}
G_1 & \xrightarrow{\ \varepsilon\ } & G_2 & \xrightarrow{\ \gamma\ } & G_3 \\
{\scriptstyle \delta}\big\uparrow & & & & \big\downarrow {\scriptstyle 0} \\
F_3 & \xleftarrow{\ \gamma'\ } & F_2 & \xleftarrow{\ \varepsilon'\ } & F_1
\end{array}
$$

be an exact sequence of abelian groups with F_1, F_2, F_3 free. Suppose that there exist row-finite matrices $A \in M_{n_1,n_1'}(\mathbb{Z})$ and $B \in M_{n_3,n_3'}(\mathbb{Z})$ for some $n_1, n_1', n_3, n_3' \in \{1, 2, \ldots, \infty\}$ with isomorphisms

$$
\alpha_1\colon \operatorname{coker} A \to G_1, \quad \beta_1\colon \ker A \to F_1,
$$

$$
\alpha_3\colon \operatorname{coker} B \to G_3, \quad \beta_3\colon \ker B \to F_3.
$$

Then there exist a row-finite matrix $Y \in M_{n_3,n_1'}(\mathbb{Z})$ and isomorphisms

$$
\alpha_2\colon \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \to G_2, \quad \beta_2\colon \ker\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} \to F_2
$$

such that the tuple $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ gives an isomorphism of complexes from the exact sequence

$$
\begin{array}{ccccc}
\operatorname{coker} A & \xrightarrow{\ I\ } & \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} & \xrightarrow{\ P\ } & \operatorname{coker} B \\
{\scriptstyle [Y]}\big\uparrow & & & & \big\downarrow {\scriptstyle 0} \\
\operatorname{coker} B & \xleftarrow{\ P'\ } & \operatorname{coker}\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix} & \xleftarrow{\ I'\ } & \operatorname{coker} A
\end{array}
$$

where the maps I, I' and P, P' are induced by the obvious inclusions or projections, to the exact sequence $\mathcal{E}$.

If there exist an $A' \in M_{n_1',n_1}(\mathbb{Z})$ such that $A'A - 1 \in M_{n_1',n_1'}(\mathbb{Z}_+)$, then Y can be chosen such that $Y \in M_{n_3,n_1'}(\mathbb{Z}_+)$. If furthermore a row-finite matrix $Z \in M_{n_3,n_1'}(\mathbb{Z})$ is given, then Y can be chosen such that $Y - Z \in M_{n_3,n_1'}(\mathbb{Z}_+)$.

As subquotients of graph algebras are graph algebras, the reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$ of a graph algebra A over X will satisfy that the group $\mathrm{K}_1(A(x))$ is

free for all $x \in X$. Combining this with the following theorem, we get at complete description of the range of reduced filtered K-theory $\mathrm{FK}_{\mathcal{R}}$.

Theorem 8.2. *Let M be an exact $\mathcal{R}$-module with $M(x_1)$ free for all $x \in X$. Then there exists a countable graph E satisfying that all vertices in E are regular and support at least two cycles, that $C^*(E)$ is tight over X and that $\mathrm{FK}_{\mathcal{R}}(C^*(E))$ is isomorphic to M. By construction $C^*(E)$ is purely infinite.*

The graph E can be chosen to be finite if (and only if) $M(x_1)$ and $M(\widetilde{x}_0)$ are finitely generated, and the rank of $M(x_1)$ coincides with the rank of the cokernel of $i\colon M(\widetilde{\partial}x_0) \to M(\widetilde{x}_0)$, for all $x \in X$. If E is chosen finite, then by construction $C^*(E)$ is a Cuntz-Krieger algebra.

Proof. For each $x \in X$, choose by [13, Proposition 3.3] a matrix $D_x \in M_{V_x}(\mathbb{Z}_+)$, where V_x is a countable, non-empty set, satisfying that $\ker D_x$ is isomorphic to $M(x_1)$ and $\mathrm{coker}\, D_x$ is isomorphic to $M(x_0) = \mathrm{coker}(M(\widetilde{\partial}x_0) \xrightarrow{i} M(\widetilde{x}_0))$, and that all vertices in the graph E_{D_x+1} are regular and support at least two cycles. If $M(x_1)$ and $M(\widetilde{x}_0)$ are finitely generated, and the rank of $M(x_1)$ coincides with the rank of the cokernel of $i\colon M(\widetilde{\partial}x_0) \to M(\widetilde{x}_0)$, then the set V_x can be chosen to be finite. Let $\varphi_{x_1}\colon M(x_1) \to \ker D_x$ and $\varphi_{x_0}\colon M(x_0) \to \mathrm{coker}\, D_x$ denote the isomorphisms.

For each $y, z \in X$ with $y \neq z$ we desire to construct a matrix $H_{yz}\colon \mathbb{Z}^{V_z} \to \mathbb{Z}^{V_y}$ with non-negative entries satisfying that H_{yz} is non-zero if and only if $y > z$, and satisfying that for each $x \in X$ there exist isomophisms $\varphi_{\widetilde{\partial}x_0}$ and $\varphi_{\widetilde{x}_0}$ making the diagrams

(8.3)

and

(8.4)

commute when $y \to x$, where $D_F \in M_{V_F}(\mathbb{Z}_+)$ for each $F \subseteq X$ is defined as

$$D_F(v,w) = \begin{cases} D_x(v,w) & v,w \in V_x \\ H_{yz}(v,w) & v \in V_y, w \in V_x, x \neq y \end{cases}$$

with $V_F = \bigcup_{y \in F} V_y$. The constructed graph E_{D_X+1} will then have the desired properties.

Let $U \in \mathbb{O}(X)$ and assume that for all $z,y \in U$, the matrices H_{yz} and isomorphisms $\varphi_{\widetilde{\partial}y_0}$ and $\varphi_{\widetilde{y}_0}$ have been defined and satisfy that the diagrams (8.3) and (8.4) commute for all $x,y \in U$ with $y \to x$. Let x be an open point in $X \setminus U$ and let us construct isomorphisms $\varphi_{\widetilde{\partial}x_0}$ and $\varphi_{\widetilde{x}_0}$, and for all $y \in \widetilde{\partial}\{x\}$ non-zero matrices H_{yx}, making the diagrams (8.3) and (8.4) commute.

Consider the commuting diagram

$$
\begin{array}{ccccccc}
\bigoplus M(\widetilde{z}_0) & \longrightarrow & \bigoplus_{y \to x} M(\widetilde{y}_0) & \longrightarrow & M(\widetilde{\partial}x_0) & \longrightarrow & 0 \\
\downarrow {\scriptstyle (\varphi_{\widetilde{z}_0})} & & \downarrow {\scriptstyle (\varphi_{\widetilde{z}_0})} & & \vdots & & \\
\bigoplus \operatorname{coker} D_{\widetilde{\{z\}}} & \longrightarrow & \bigoplus_{y \to x} \operatorname{coker} D_{\widetilde{\{y\}}} & \longrightarrow & \operatorname{coker} D_{\widetilde{\partial}\{x\}} & \longrightarrow & 0.
\end{array}
$$

The top row is exact by exactness of M, and the bottom row is exact by exactness of $\mathrm{FK}(C^*(E_{D_{\widetilde{\partial}\{x\}}+1}))$. An isomorphism $\varphi_{\widetilde{\partial}x_0}\colon M(\widetilde{\partial}x_0) \to \operatorname{coker} D_{\widetilde{\partial}\{x\}}$ is therefore induced. By construction, (8.4) commutes for all $y \to x$.

Now consider the commuting diagram

where a free group F and maps into and out of it have been chosen so that the inner six-term sequence is exact. Apply Theorem 8.1 to the inner six-term exact sequence to get non-zero matrices H_{yx} for all $y \in \widetilde{\partial}\{x\}$ realizing the sequence, i.e., making (8.3) commute. $\qquad\square$

Corollary 8.5. *Let A be a C^*-algebra of real rank zero over X, and assume that $\mathrm{K}_1(A(x))$ is free for all $x \in X$. Then there exists a purely infinite graph algebra $C^*(E)$ that is tight over X and has $\mathrm{FK}_{\mathcal{R}}(C^*(E)) \cong \mathrm{FK}_{\mathcal{R}}(A)$. If X is an EBP space, then automatically $\mathrm{FK}_{\mathcal{ST}}(C^*(E)) \cong \mathrm{FK}_{\mathcal{ST}}(A)$.*

If furthermore for all $x \in X$, the group $\mathrm{K}_*(A(x))$ is finitely generated and $\operatorname{rank}\mathrm{K}_1(A(x)) = \operatorname{rank}\mathrm{K}_0(A(x))$, then $C^*(E)$ can be chosen to be a purely infinite Cuntz-Krieger algebra.

9. Unital filtered K-theory

In [23, 2.1], Gunnar Restorff and Efren Ruiz showed that if a functor F (that includes K_0) strongly classifies a certain class of C^*-algebra s up to stable isomorphism, then the functor $(F, [1_-])$ with the class of the unit in K_0 added, classifies unital, properly infinite C^*-algebras in the class up to unital isomorphism. A version with slightly generalized assumptions of this so-called *meta-theorem* may be found in [15]. With these generalized assumptions, the theorem applies to filtered K-theory FK over accordion spaces X with respect to Kirchberg X-algebras with simple subquotients in the bootstrap class.

Let X be an arbitrary finite T_0-space. For $x, x' \in X$, we let $\inf(x, x')$ denote the set $\{y \in X \mid y \to x, y \to x'\}$.

Definition 9.1. The category $\mathfrak{Mod}(\mathcal{ST})^{\mathrm{pt}}$ of *pointed $\mathcal{ST}$-modules* is defined to have objects (M, m) where M is a $\mathcal{ST}$-module and $m \in M(X, 0)$, and morphisms $\varphi \colon (M, m) \to (N, n)$ that are $\mathcal{ST}$-morphisms with $\varphi(m) = n$.

The category $\mathfrak{Mod}(\mathcal{B})^{\mathrm{pt}}$ of *pointed $\mathcal{B}$-modules* is defined similarly with objects (M, m) where M is a $\mathcal{B}$-module and

$$
m \in \mathrm{coker}\left(\bigoplus_{y \in \inf(x, x')} M(\widetilde{y}_0) \xrightarrow{\left(i_{\widetilde{y}_0}^{\widetilde{x}_0} \quad -i_{\widetilde{y}_0}^{\widetilde{x}'_0} \right)_{y \in \inf(x, x')}} \bigoplus_{x \in X} M(\widetilde{x}_0, 0) \right),
$$

and a morphism $\varphi \colon (M, m) \to (N, n)$ is a $\mathcal{B}$-morphism whose induced map on the cokernels sends m to n.

Similarly, the categories $\mathfrak{Mod}(\mathcal{BR})^{\mathrm{pt}}$ and $\mathfrak{Mod}(\mathcal{R})^{\mathrm{pt}}$ of *pointed $\mathcal{BR}$-modules* respectively *pointed $\mathcal{R}$-modules* are defined.

Definition 9.2. A pointed $\mathcal{ST}$-module (M, m) is called *exact* if M is an exact $\mathcal{ST}$-module, and *real-rank-zero-like* if M is real-rank-zero-like. Similary, a pointed $\mathcal{B}$-module, $\mathcal{BR}$-module, or $\mathcal{R}$-module (M, m) is called *exact* if M is exact.

Lemma 9.3. *Let M be an exact real-rank-zero-like $\mathcal{ST}$-module. Then the sequence*

$$
\bigoplus_{y \in \inf(x, x')} M(\widetilde{\{y\}}, 0) \xrightarrow{\left(i_{\{y\}}^{\widetilde{\{x\}}} \quad -i_{\{y\}}^{\widetilde{\{x\}}'} \right)_{y \in \inf(x, x')}} \bigoplus_{x \in X} M(\widetilde{\{x\}}, 0) \xrightarrow{(i_{\{x\}}^{X})_{x \in X}} M(X, 0) \to 0
$$

is exact.

Proof. By Lemma 6.8 the horizontal row of the following commuting diagram is exact:

$$
\bigoplus_{x, x' \in X} M(\widetilde{\{x\}} \cap \widetilde{\{x'\}}, 0) \xrightarrow{\left(i_{\{x\} \cap \{x'\}}^{\widetilde{\{x\}}} \quad -i_{\{x\} \cap \{x'\}}^{\widetilde{\{x\}}'} \right)_{x' \in X}^{x \in X}} \bigoplus_{x \in X} M(\widetilde{\{x\}}, 0) \longrightarrow M(X, 0) \longrightarrow 0
$$

Furthermore, since for any pair $x, x' \in X$ the collection $(\widetilde{\{y\}})_{y \in \inf(x, x')}$ covers $\widetilde{\{x\}} \cap \widetilde{\{x'\}}$, we see by Lemma 6.8 that the vertical map in the diagram is surjective. This establishes the desired result. $\square$

Definition 9.4. Let A be a unital C^*-algebra over a finite T_0-space X. Its *unital concrete filtered* K-*theory* $\mathrm{FK}_{\mathcal{ST}}^{\mathrm{unit}}(A)$ is defined as the pointed $\mathcal{ST}$-module $(\mathrm{FK}_{\mathcal{ST}}(A), [1_A])$.

If A has real rank zero, then its *unital reduced filtered* K-*theory* $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(A)$ is defined as the pointed $\mathcal{R}$-module $(\mathrm{FK}_{\mathcal{R}}(A), u)$ where u is the unique element in

$$
\operatorname{coker}\left(\bigoplus_{y\in\inf(x,x')}\mathrm{FK}_{\widetilde{\{y\}}}^0(A) \xrightarrow{\left(i_{\widetilde{y_0}}^{\widetilde{x}_0} \quad -i_{\widetilde{y_0}}^{\widetilde{x}_0'}\right)_{y\in\inf(x,x')}} \bigoplus_{x\in X}\mathrm{FK}_{\widetilde{\{x\}}}^0(A)\right)
$$

that is mapped to $[1_A]$ in $\mathrm{K}_0(A)$ by the map induced by the family $(\mathrm{FK}_{\widetilde{\{x\}}}^0(A) \xrightarrow{i_{\widetilde{\{x\}}}^X}$ $\mathrm{FK}_X^0(A))_{x\in X}$, cf. Lemma 9.3.

If A has real rank zero and X is a UP space, then its *unital filtered* K-*theory restricted to canonical base* $\mathrm{FK}_{\mathcal{B}}^{\mathrm{unit}}(A)$ is defined similarly.

Theorem 9.5. *For any EBP space X, the functor $\mathfrak{F}_{\mathcal{B}}\colon \mathfrak{Mod}(\mathcal{ST}) \to \mathfrak{Mod}(\mathcal{B})$ induces an equivalence of categories between pointed, exact, real-rank-zero-like $\mathcal{ST}$-modules and pointed exact $\mathcal{B}$-modules.*

Proof. Let (N, n) be an exact pointed $\mathcal{B}$-module, and let M be the exact real-rank-zero-like $\mathcal{ST}$-module constructed in the proof of Theorem 5.17 such that $\mathfrak{F}_{\mathcal{B}}(M) = N$. Since M is exact and real-rank-zero-like, Lemma 9.3 guarantees the unique existence of an element m in $M(X, 0)$ that is mapped to n. So the functor G constructed in the proof of Theorem 5.17 induces a functor from exact pointed $\mathcal{B}$-modules to exact pointed real-rank-zero-like $\mathcal{ST}$-modules, and the desired follows. $\qquad\square$

Proposition 9.6. *Assume that X is a UP space. Let (M, m) and (N, n) be exact pointed $\mathcal{BR}$-modules with $M(x_1)$ and $N(x_1)$ free for all non-open points $x \in X$, and let $\varphi\colon \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(M) \to \mathfrak{F}_{\mathcal{BR},\mathcal{R}}(N)$ be a pointed $\mathcal{R}$-module homomorphism. Then there exists a (not necessarily unique) pointed $\mathcal{BR}$-module homomorphism $\Phi\colon M \to N$ satisfying $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}(\Phi) = \varphi$, and if φ is an isomorphism, then Φ is by construction an isomorphism.*

Proof. This follows from Theorem 7.8 since the groups $M(\widetilde{x}_0)$ are not forgotten by $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}$. $\qquad\square$

Corollary 9.7. *Let X be an accordion space, and let A and B be unital Kirchberg X-algebras of real rank zero with all simple subquotients in the bootstrap class. If $\mathrm{FK}_{\mathcal{B}}^{\mathrm{unit}}(A) \cong \mathrm{FK}_{\mathcal{B}}^{\mathrm{unit}}(B)$, then $A \cong B$.*

Proof. This follows the meta-theorem in [15] together with Theorem 3.11 and Corollary 5.19 respectively. $\qquad\square$

Corollary 9.8. *Let X be an accordion space, and let A and B be unital Kirchberg X-algebras of real rank zero with all simple subquotients in the bootstrap class. Assume that $\mathrm{K}_1(A(x))$ and $\mathrm{K}_1(B(x))$ are free groups for all $x \in X$. If $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(A) \cong \mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(B)$, then $A \cong B$.*

Proof. This follows the meta-theorem in [15] together with Theorem 3.11 and Corollary 7.14. $\qquad\square$

Remark 9.9. In [2] the first named author, Gunnar Restorff, and Efren Ruiz studied four four-point spaces that are connected UP spaces but not accordion spaces. For these spaces, $\mathcal{NT}$ and $\mathcal{ST}$ coincide, and they showed that if X is one of these spaces, then FK is a complete invariant for stable Kirchberg X-algebras of real rank zero. Therefore, Corollaries 9.7 and 9.8 also hold for these spaces.

Furthermore, the proof of Proposition 7.16 also applies to $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}$ and unital C^*-algebras, hence Corollary 9.8 also holds for the space $\mathcal{D}$.

Recall that for a unital graph algebra $C^*(E)$, the class of the unit $[1_{C^*(E)}]$ in $\mathrm{K}_0(C^*(E))$ is sent via the canonical isomorphism $\mathrm{K}_0(C^*(E)) \to \operatorname{coker} D_E$ to the class $[\mathbb{1}] = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} + \operatorname{im} D_E$, when $1 + D_E$ denotes the adjacency matrix for E.

Theorem 9.10. *Let X be a finite T_0-space, and let (M, m) be an exact pointed $\mathcal{R}$-module. Assume that for all $x \in X$, $M(x_1)$ is a free group,*

$$\operatorname{coker}(M(\widetilde{\partial}x_0) \xrightarrow{\,i_{\widetilde{\partial}x_0}^{\widetilde{x}_0}\,} M(\widetilde{x}_0))$$

is finitely generated, and $\operatorname{rank} M(x_1) \leq \operatorname{rank} \operatorname{coker}(M(\widetilde{\partial}x_0) \xrightarrow{\,i_{\widetilde{\partial}x_0}^{\widetilde{x}_0}\,} M(\widetilde{x}_0))$.

Then there exists a countable graph E satisfying that all vertices in E support at least two cycles, that E^0 is finite, that $C^*(E)$ is tight over X, and that $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(C^*(E))$ is isomorphic to (M, m). By construction $C^*(E)$ is unital and purely infinite.

The graph E can be chosen to have regular vertices if (and only if) the rank of $M(x_1)$ coincides with the rank of the cokernel of $i \colon M(\widetilde{\partial}x_0) \to M(\widetilde{x}_0)$ for all $x \in X$. If E is chosen to have regular vertices, then by construction $C^*(E)$ is a Cuntz-Krieger algebra.

Proof. By Theorem 8.2 there exists a C^*-algebra A of real rank zero with $\mathrm{FK}_{\mathcal{R}}(A) \cong M$. So we may consider a real-rank-zero-like exact pointed $\mathcal{ST}$-module (M, m) and aim to construct $C^*(E)$ such that $\mathfrak{F}_{\mathcal{R}}(M, m) = \mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(C^*(E))$. Notice that, as in the proof of Theorem 5.17, by using the cosheaf condition any isomorphism $\mathfrak{F}_{\mathcal{R}}(M) \to \mathrm{FK}_{\mathcal{R}}(C^*(E))$ may be uniquely extended to an isomorphism $M(Y, 0) \to \mathrm{FK}_Y^0(C^*(E))$ for all locally closed subsets Y of X.

The construction is similar to the construction in the proof of Theorem 8.2. Let x_1 be an open point in X and define $U_1 = \{x_1\}$. Define U_k recursively by choosing an open point x_k in $X \setminus U_{k-1}$ and defining $U_k = U_{k-1} \cup \{x_k\}$. Let C_k denote the largest subset of U_k that is closed in X. Observe that

$$C_k = X \setminus \bigcup_{y \in \mathrm{CP}(X) \setminus U_k} \widetilde{\{y\}}$$

where $\mathrm{CP}(X)$ denotes the closed points in X. And observe that if x_k is closed in X then $C_k \setminus C_{k-1} \subseteq \widetilde{\{x_k\}}$ and $C_k \setminus \widetilde{\{x_k\}} = C_{k-1}$, and otherwise $C_k = C_{k-1}$. Define for all closed subsets C of X the element m_C of $M(C, 0)$ as the image of m under $r \colon M(X, 0) \to M(C, 0)$.

For each x not closed in X, choose by Proposition 3.6 of [13] a graph E_x that is transitive, has finitely many vertices that all support at least two cycles, and such that $\mathrm{K}_1(C^*(E_x))$ is isomorphic to $M(\{x\}, 1)$ and $\mathrm{K}_0(C^*(E_x))$ isomorphic to $M(\{x\}, 0)$. Define $V_{\{x\}} = E_x^0$ and $V_{\{x\}}' = (E_x^0)_{\mathrm{reg}}$, let $D_{\{x\}} \in M_{V_{\{x\}}}(\mathbb{Z}_+ \cup \{\infty\})$

such that $1 + D_{\{x\}}$ is the adjacency matrix for E_x, and let $D'_{\{x\}}$ denote the $V'_{\{x\}} \times V_{\{x\}}$ matrix defined by $D'_{\{x\}}(v, w) = D_{\{x\}}(v, w)$. If $\operatorname{rank} M(\{x\}, 1) = \operatorname{rank} M(\{x\}, 0)$ then $V_{\{x\}} = V'_{\{x\}}$. Let isomorphisms $\varphi^1_{\{x\}} \colon M(\{x\}, 1) \to \ker D'_{\{x\}}$ and $\varphi^0_{\{x\}} \colon M(\{x\}, 0) \to \operatorname{coker} D'_{\{x\}}$ be given. For x closed in X we may by Proposition 3.6 of [13] choose E_x and $\varphi^0_{\{x\}}$ such that furthermore $\varphi^0_{\{x\}}(m_{\{x\}}) = [\mathbb{1}]$.

As in the proof of Theorem 8.2 we wish to construct a finite nonempty set V_X and a matrix $D_X \in M_{V_X}(\mathbb{Z}_+ \cup \{\infty\})$ such that E_{1+D_X} is the desired graph. We do so by for each $U \in \mathbb{O}(X)$ defining $V_U = \bigcup_{x \in U} V_{\{x\}}$ and $V'_U = \bigcup_{x \in U} V'_{\{x\}}$, and by for each $x \in X$ constructing suitable submatrices $D_{\widetilde{\{x\}}} \in M_{V_{\widetilde{\{x\}}}}(\mathbb{Z}_+ \cup \{\infty\})$ and $D'_{\widetilde{\{x\}}} \in M_{V'_{\widetilde{\{x\}}}, V_{\widetilde{\{x\}}}}(\mathbb{Z}_+)$ of D_X with $D_{\widetilde{\{x\}}}(v, w) = D'_{\widetilde{\{x\}}}(v, w)$ when $v \in V'_{\widetilde{\{x\}}}$ and $D_{\widetilde{\{x\}}}(v, w) = \infty$ otherwise. These will be constructed recursively over $k \in \{1, \ldots, n\}$ to satisfy the following: For each $x \in X$, there are isomorphisms making the diagram

(9.11)
$$
\begin{array}{ccccc}
M(\{x\}, 1) & \xrightarrow{\ \delta\ } & M(\widetilde{\partial}\{x\}, 0) & \xrightarrow{\ i\ } & M(\widetilde{\{x\}}, 0) \\
\cong \downarrow \varphi^1_{\{x\}} & & \cong \downarrow \varphi^0_{\widetilde{\partial}\{x\}} & & \cong \downarrow \varphi^0_{\widetilde{\{x\}}} \\
\ker D'_{\{x\}} & \xrightarrow{\ \delta\ } & \operatorname{coker} D'_{\widetilde{\partial}\{x\}} & \xrightarrow{\ i\ } & \operatorname{coker} D'_{\widetilde{\{x\}}}
\end{array}
$$

commute and satisfying for all $y \to x$ that

(9.12)
$$
\begin{array}{ccc}
M(\widetilde{\{y\}}, 0) & \xrightarrow{\ i\ } & M(\widetilde{\partial}\{x\}, 0) \\
\cong \downarrow \varphi^0_{\widetilde{\{y\}}} & & \cong \downarrow \varphi^0_{\widetilde{\partial}\{x\}} \\
\operatorname{coker} D'_{\widetilde{\{y\}}} & \xrightarrow{\ i\ } & \operatorname{coker} D'_{\widetilde{\partial}\{x\}}
\end{array}
$$

commutes, and for all $k \in \{1, \ldots, n\}$ that the isomorphism

$$\varphi^0_{C_k} \colon M(C_k, 0) \to \operatorname{coker} D_{C_k}$$

induced by $(\varphi^0_{\widetilde{\{x\}}})_{x \in X}$ sends m_{C_k} to $[\mathbb{1}]$.

Assume that x_k is a closed point in X. Since $D_{C_k \setminus \{x_k\}}$ is already defined, we may define $1_{C_k \setminus \widetilde{\{x_k\}}}$ as the element in $\mathbb{Z}^{V_{C_k \setminus \{x_k\}}}$ with

$$
1_{C_k \setminus \widetilde{\{x_k\}}}(i) = \begin{cases} 1 & \text{if } i \in V_{C_k \setminus \widetilde{\{x_k\}}} \\ 0 & \text{if } i \in V_{(C_k \cap \widetilde{\{x_k\}}) \setminus \{x_k\}}. \end{cases}
$$

Define $\widetilde{m}_{C_k \setminus \{x_k\}}$ as the preimage of $1_{C_k \setminus \widetilde{\{x_k\}}}$ under the isomorphism $\varphi^0_{C_k \setminus \{x_k\}}$ induced by $(\varphi^0_{\widetilde{\{y\}}})_{y \in U_{k-1}}$. Notice that $\varphi^0_{C_{k-1}}(m_{C_k} r^{C_{k-1}}_{C_k}) = [\mathbb{1}]$ and that, since C_{k-1} is closed in $C_k \setminus \{x_k\}$,

$$
\begin{aligned}
\varphi^0_{C_{k-1}}(\widetilde{m}_{C_k \setminus \{x_k\}} i^{C_k}_{C_k \setminus \{x_k\}} r^{C_{k-1}}_{C_k}) &= \varphi^0_{C_{k-1}}(\widetilde{m}_{C_k \setminus \{x_k\}} r^{C_{k-1}}_{C_k \setminus \{x_k\}}) \\
&= \varphi^0_{C_k \setminus \{x_k\}}(\widetilde{m}_{C_k \setminus \{x_k\}}) r^{C_{k-1}}_{C_k \setminus \{x_k\}} \\
&= 1_{C_k \setminus \widetilde{\{x_k\}}} r^{C_{k-1}}_{C_k \setminus \{x_k\}} = [\mathbb{1}],
\end{aligned}
$$

so by injectivity of $\varphi^0_{C_{k-1}}$, $m_{C_k} - \widetilde{m}_{C_k \setminus \{x_k\}} i^{C_k}_{C_k \setminus \{x_k\}}$ lies in $\ker r^{C_{k-1}}_{C_k} = \operatorname{im} i^{C_k}_{C_k \cap \widetilde{\{x_k\}}}$.
Choose $\widetilde{m}_{C_k \cap \widetilde{\{x_k\}}}$ in $M(C_k \cap \widetilde{\{x_k\}}, 0)$ such that

$$m_{C_k} = m_{C_k \cap \widetilde{\{x_k\}}} i^{C_k}_{C_k \cap \widetilde{\{x_k\}}} + \widetilde{m}_{C_k \setminus \{x_k\}} i^{C_k}_{C_k \setminus \{x_k\}}.$$

Choose $\widetilde{m}_{\widetilde{\{x_k\}}}$ in $M(\widetilde{\{x_k\}}, 0)$ such that

$$\widetilde{m}_{\widetilde{\{x_k\}}} r^{C_k \cap \widetilde{\{x_k\}}}_{\widetilde{\{x_k\}}} = \widetilde{m}_{C_k \cap \widetilde{\{x_k\}}}.$$

Recall from the proof of Theorem 8.2 that $(\varphi^0_{\widetilde{\{y\}}})_{y \in \widetilde{\partial}\{x_k\}}$ induces an isomorphism $\varphi^0_{\widetilde{\partial}\{x_k\}}$. Consider

$$(9.13) \qquad \begin{array}{ccccccc}
M(\{x_k\}, 1) & \xrightarrow{\delta} & M(\widetilde{\partial}\{x_k\}, 0) & \xrightarrow{i} & M(\widetilde{\{x_k\}}, 0) & \xrightarrow{r} & M(\{x_k\}, 0) \\
\cong \downarrow \varphi^1_{\{x_k\}} & & \cong \downarrow \varphi^0_{\widetilde{\partial}\{x_k\}} & & & & \cong \downarrow \varphi^0_{\{x_k\}} \\
\ker D'_{x_k} & \longrightarrow & \operatorname{coker} D'_{\widetilde{\partial}\{x_k\}} & & & & \operatorname{coker} D'_{\{x_k\}}
\end{array}$$

where $\widetilde{m}_{\widetilde{\{x_k\}}}$ is mapped to $m_{\{x_k\}}$ which by $\varphi^0_{\{x_k\}}$ is mapped to $[\mathbb{1}]$, and use [13, Proposition 4.8] to realise (9.13) as $D'_{\widetilde{\{x_k\}}}$ via $\varphi^0_{\widetilde{\{x_k\}}} : M(\widetilde{\{x_k\}}, 0) \to \operatorname{coker} D'_{\widetilde{\{x_k\}}}$ that sends $\widetilde{m}_{\widetilde{\{x_k\}}}$ to $[\mathbb{1}]$. Define $D_{\widetilde{\{x_k\}}} \in M_{V_{\widetilde{\{x_k\}}}}(\mathbb{Z}_+ \cup \{\infty\})$ by $D_{\widetilde{\{x_k\}}}(v, w) = D'_{\widetilde{\{x_k\}}}(v, w)$ when $v \in V'_{\widetilde{\{x_k\}}}$ and by $D_{\widetilde{\{x_k\}}}(v, w) = \infty$ when $v \in V_{\widetilde{\{x_k\}}} \setminus V'_{\widetilde{\{x_k\}}}$.

That (9.11) and (9.12) hold for x_k follows immediately from the construction. To verify that the map $\varphi^0_{C_k}$ induced by $(\varphi^0_{\widetilde{\{y\}}})_{y \in U_k}$ satisfies $\varphi^0_{C_k}(m_{C_k}) = [\mathbb{1}]$, observe that the map $\varphi^0_{C_k \cap \widetilde{\{x_k\}}}$ induced by $\varphi^0_{\widetilde{\{x_k\}}}$ will map $\widetilde{m}_{C_k \cap \widetilde{\{x_k\}}}$ to $[\mathbb{1}]$, and consider the commuting diagram

$$\begin{array}{ccc}
M(C_k \setminus \{x_k\}, 0) \oplus M(C_k \cap \widetilde{\{x_k\}}) & \xrightarrow{\binom{i}{i}} & M(C_k, 0) \\
\cong \downarrow \varphi^0_{C_k \setminus \{x_k\}} \oplus \varphi^0_{C_k \cap \widetilde{\{x_k\}}} \quad \binom{i}{i} & & \cong \downarrow \varphi^0_{C_k} \\
\operatorname{coker} D'_{C_k \setminus \{x_k\}} \oplus \operatorname{coker} D'_{C_k \cap \widetilde{\{x_k\}}} & \xrightarrow{\binom{i}{i}} & \operatorname{coker} D'_{C_k}.
\end{array}$$

Since $(\widetilde{m}_{C_k \setminus \{x_k\}}, \widetilde{m}_{C_k \cap \widetilde{\{x_k\}}})$ is mapped to m_{C_k} by $\binom{i}{i}$ and to $(1_{C_k \setminus \widetilde{\{x_k\}}}, [\mathbb{1}])$ by $\varphi^0_{C_k \setminus \{x_k\}} \oplus \varphi^0_{C_k \cap \widetilde{\{x_k\}}}$, we see by commutativity of the diagram that $\varphi^0_{C_k}(m_{C_k}) = [\mathbb{1}]$.

For k with x_k not closed in X, C_k equals C_{k-1} and a construction similar to the one in the proof of Theorem 8.2 applies. As in the proof of Theorem 8.2, Proposition 4.7 of [13] allows us to make sure that E_{D_X+1} will be tight over X. $\qquad\square$

Corollary 9.14. *Let X be a finite T_0-space and let A be a unital C^*-algebra of real rank zero over X. Assume for all $x \in X$ that $\mathrm{K}_1(A(x))$ is free, $\mathrm{K}_0(A(x))$ is finitely generated, and $\operatorname{rank} \mathrm{K}_1(A(x)) \leq \operatorname{rank} \mathrm{K}_0(A(x))$.*

Then there exists a countable graph E for which $C^*(E)$ is unital, purely infinite, and tight over X such that $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(C^*(E)) \cong \mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(A)$. If X is an EBP space, then automatically $\mathrm{FK}_{\mathcal{ST}}^{\mathrm{unit}}(C^*(E)) \cong \mathrm{FK}_{\mathcal{ST}}^{\mathrm{unit}}(A)$.

If furthermore $\mathrm{rank}\, \mathrm{K}_1(A(x)) = \mathrm{rank}\, \mathrm{K}_0(A(x))$ for all $x \in X$, then E can be chosen such that $C^*(E)$ is a purely infinite Cuntz-Krieger algebra.

Corollary 9.15. *Let X be an accordion space, and let $I \hookrightarrow A \twoheadrightarrow B$ be an extension of C^*-algebras. Assume that A is unital and tight over X.*

Then A is a Cuntz-Krieger algebra of real rank zero if and only if

- *I is stably isomorphic to a Cuntz-Krieger algebra of real rank zero,*
- *B is a Cuntz-Krieger algebra of real rank zero,*
- *and the exponential map $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes.*

Proof. Assume that A is a Cuntz-Krieger algebra of real rank zero. It is well-known that then B is also a Cuntz-Krieger algebra of real rank zero and I is stably isomorphic to one. By Theorem 4.2 of [20], $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes since A has real rank zero and therefore is K_0-liftable, cf. Remark 3.10.

Now, assume that B is a Cuntz-Krieger algebra of real rank zero, that I is stably isomorphic to one, and that the map $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes. By Theorem 4.3 of [26], A is O_∞-absorbing since B and I are. Since B and I are K_0-liftable and $\mathrm{K}_0(B) \to \mathrm{K}_1(I)$ vanishes, A is also K_0-liftable (i.e., $\mathrm{FK}(A)$ is real-rank-zero-like) by [6, Proposition 3.5]. So by pure infiniteness of A it therefore follows from Theorem 4.2 of [20] that A has real rank zero. For all $x \in X$, $\mathrm{K}_1(A(x))$ is free since B and I are stably isomorphic to Cuntz-Krieger algebras. So by Theorem 9.10 there exists a real rank zero Cuntz-Krieger algebra C that is tight over X and has $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(A) \cong \mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}(C)$. By Corollary 9.8, A and C are isomorphic. $\square$

Remark 9.16. Corollary 9.15 holds for spaces X for which $\mathrm{FK}_{\mathcal{R}}^{\mathrm{unit}}$ is a complete invariant for unital Kirchberg X-algebras A where $A(x)$ is in the bootstrap class and $\mathrm{K}_1(A(x))$ is free for all $x \in X$, cf. Remark 9.9.

10. Ordered filtered K-theory

The notion of ordered filtered K-theory was introduced by Søren Eilers, Gunnar Restorff, and Efren Ruiz in [14] to classify (not necessarily purely infinite) graph algebras of real rank zero.

Recall that for a C^*-algebra A, a class $[p]_0$ in $\mathrm{K}_0(A)$ where p is a projection in $M_n(A)$ for some n is called *positive*. The *positive cone* $\mathrm{K}_0(A)^+$ consists of all positive elements in $\mathrm{K}_0(A)$. For two C^*-algebras A and B, a group homomorphism $\varphi\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ is called *positive* if $\varphi(\mathrm{K}_0(A)^+) \subseteq \mathrm{K}_0(B)^+$, and a group isomorphism $\varphi\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ is called an *order-isomorphism* if $\varphi(\mathrm{K}_0(A)^+) = \mathrm{K}_0(B)^+$.

Note that for a finite topological space X, a locally closed subset Y of X, and an open subset U of Y, the maps $i_U^Y\colon \mathrm{K}_0(A(U)) \to \mathrm{K}_0(A(Y))$ and $r_Y^{Y \setminus U}\colon \mathrm{K}_0(A(Y)) \to \mathrm{K}_0(A(Y \setminus U))$ are positive.

Definition 10.1. For C^*-algebras A and B over a finite topological space X, an $\mathcal{ST}$-module homomorphism $\varphi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ is called *positive* if the induced maps $\mathrm{FK}_Y^0(A) \to \mathrm{FK}_Y^0(B)$ are positive for all $Y \in \mathbb{LC}(X)$, and an $\mathcal{ST}$-module isomorphism $\mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ is called an *order-isomorphism* if the induced isomomorphisms are order-isomorphisms. For the reductions $\mathrm{FK}_{\mathcal{R}}$, $\mathrm{FK}_{\mathcal{B}}$, and $\mathrm{FK}_{\mathcal{BR}}$ of filtered K-theory, analogous definitions apply.

We are indebted to Mikael Rørdam for the elegant proof of the following lemma.

Lemma 10.2. *Let A be a real-rank-zero C^*-algebra and let I and J be (closed, two-sided) ideals in A satisfying $I + J = A$. Then any projection p in A can be written $p = q + q'$ with q a projection in I and q' a projection in J.*

Proof. Let p a projection in A be given and write $p = a + b$ with $a \in I$ and $b \in J$. We may assume that $a = pap$ and $b = pbp$. As A has real rank zero, the hereditary subalgebra pIp has an approximate unit of projections, so there exists a projection q in pIp satisfying $\|a - aq\| < 1$. Since $q = pqp$, $q \leq p$ and we may define a projection q' as $q' = p - q$. It remains to prove $q' \in J$. We have

$$\|q' - q'bq'\| = \|q'(p - b)q'\| = \|q'a(p - q)\| \leq \|q'\|\|a - aq\| < 1.$$

Since $q'bq' \in J$, the image of q' in the quotient A/J is a projection of norm strictly less than 1. Since such a projection is 0, we get $q' \in J$. □

The following theorem is a version of Corollary 5.19 taking the order into account.

Theorem 10.3. *Let X be an EBP space, and let A and B be C^*-algebras over X of real rank zero. Then for any order-isomorphism $\varphi\colon \mathrm{FK}_{\mathcal{B}}(A) \to \mathrm{FK}_{\mathcal{B}}(B)$ there is a unique order-isomorphism $\Phi\colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ satisfying $\mathfrak{F}_{\mathcal{B}}(\Phi) = \varphi$.*

Proof. By Corollary 5.19, Φ is an isomorphism if and only if φ is. Assume that φ is an order-isomorphism, and let us show first for $Y \in \mathbb{O}(X)$ and then for $Y \in \mathbb{LC}(X)$ that Φ_Y^0 is an order-isomorphism.

For U an open subset of X, the following diagram has commuting squares and its rows are exact by Lemmas 4.7 and 4.8.

$$
\begin{array}{ccccccc}
\bigoplus_{y\in\inf(x,x')} \mathrm{FK}_{\widetilde{\{y\}}}^0(A) & \xrightarrow{\left(i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}} \; -i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}\,'}\right)} & \bigoplus_{x\in U} \mathrm{FK}_{\widetilde{\{x\}}}^0(A) & \xrightarrow{(i_{\widetilde{\{x\}}}^U)} & \mathrm{FK}_U^0(A) & \longrightarrow & 0 \\
\downarrow{\scriptstyle (\varphi_{\widetilde{\{y\}}}^0)} & & \downarrow{\scriptstyle (\varphi_{\widetilde{\{x\}}}^0)} & & \downarrow{\scriptstyle \Phi_U^0} & & \\
\bigoplus_{y\in\inf(x,x')} \mathrm{FK}_{\widetilde{\{y\}}}^0(B) & \xrightarrow{\left(i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}} \; -i_{\widetilde{\{y\}}}^{\widetilde{\{x\}}\,'}\right)} & \bigoplus_{x\in U} \mathrm{FK}_{\widetilde{\{x\}}}^0(B) & \xrightarrow{(i_{\widetilde{\{x\}}}^U)} & \mathrm{FK}_U^0(B) & \longrightarrow & 0
\end{array}
$$

Since $(A(\widetilde{\{x\}}))_{x\in U}$ is a finite collection of ideals in $A(U)$, we see by Lemma 10.2 that the map $(i_{\widetilde{\{x\}}}^U)\colon \bigoplus_{x\in U} \mathrm{K}_0(A(\widetilde{\{x\}}) \to \mathrm{K}_0(A(U))$ surjects $\bigoplus_{x\in U} \mathrm{K}_0(A(\widetilde{\{x\}})^+$ onto $\mathrm{K}_0(A(U))^+$. Similarly, the map $(i_{\widetilde{\{x\}}}^U)\colon \bigoplus_{x\in U} \mathrm{K}_0(B(\widetilde{\{x\}})) \to \mathrm{K}_0(B(U))$ surjects $\bigoplus_{x\in U} \mathrm{K}_0(B(\widetilde{\{x\}}))^+$ onto $\mathrm{K}_0(B(U))^+$. A simple diagram chase therefore shows that Φ_U^0 is an order-isomorphism since the map $\varphi_{\widetilde{\{x\}}}^0$ is an order-isomorphism for all $x \in U$.

For a locally closed subset Y of X, choose open subsets U and V of X satisfying $V \subseteq U$ and $U \setminus V = Y$. Then Φ_U^0 is an order-isomorphism. Consider the following diagram with exact rows and commuting squares.

$$
\begin{array}{ccccccc}
\mathrm{FK}_V^0(A) & \xrightarrow{i_V^U} & \mathrm{FK}_U^0(A) & \xrightarrow{r_U^Y} & \mathrm{FK}_Y^0(A) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \Phi_V^0} & & \downarrow{\scriptstyle \Phi_U^0} & & \downarrow{\scriptstyle \Phi_Y^0} & & \\
\mathrm{FK}_V^0(B) & \xrightarrow{i_V^U} & \mathrm{FK}_U^0(B) & \xrightarrow{r_U^Y} & \mathrm{FK}_Y^0(B) & \longrightarrow & 0
\end{array}
$$

In [10, Theorem 3.14], Lawrence G. Brown and Gert K. Pedersen showed that given an extension $I \hookrightarrow C \twoheadrightarrow C/I$ of C^*-algebras, the C^*-algebra C has real rank zero if and only if I and C/I have real rank zero and projections in C/I lift to projections in C. Thus, since A and therefore $M_n \otimes A(U)$ for all n has real rank zero, the map $i_U^Y \colon \mathrm{K}_0(A(U)) \to \mathrm{K}_0(A(Y))$ surjects $\mathrm{K}_0(A(U))^+$ onto $\mathrm{K}_0(A(Y))^+$. Similarly, the map $i_U^Y \colon \mathrm{K}_0(B(U)) \to \mathrm{K}_0(B(Y))$ surjects $\mathrm{K}_0(B(U))^+$ onto $\mathrm{K}_0(B(Y))^+$. A simple diagram chase therefore shows that Φ_Y^0 is an order-isomorphism. $\qquad\square$

We have the following ordered analogous of Theorem 7.8 and Corollary 7.15.

Theorem 10.4. *Let X be a UP space, and let A and B be C^*-algebras over X of real rank zero. Assume that $\mathrm{K}_1(A(\{x\}))$ and $\mathrm{K}_1(B(\{x\}))$ are free groups for all non-open points $x \in X$. Then for any order-isomorphism $\varphi \colon \mathrm{FK}_\mathcal{R}(A) \to \mathrm{FK}_\mathcal{R}(B)$ there exists a (not necessarily unique) order-isomorphism $\Phi \colon \mathrm{FK}_{\mathcal{BR}}(A) \to \mathrm{FK}_{\mathcal{BR}}(B)$ that satisfies $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}(\Phi) = \varphi$.*

Proof. Since the functor $\mathfrak{F}_{\mathcal{BR},\mathcal{R}}$ only forgets K_1-groups, the desired follows immediately from Theorem 7.8. $\qquad\square$

Corollary 10.5. *Let X be an EBP space, and let A and B be C^*-algebras over X of real rank zero. Assume that $\mathrm{K}_1(A(\{x\}))$ and $\mathrm{K}_1(B(\{x\}))$ are free groups for all $x \in X$. Then for any order-isomorphism $\varphi \colon \mathrm{FK}_\mathcal{R}(A) \to \mathrm{FK}_\mathcal{R}(B)$ there exists a order-isomorphism $\Phi \colon \mathrm{FK}_{\mathcal{ST}}(A) \to \mathrm{FK}_{\mathcal{ST}}(B)$ that satisfies $\mathfrak{F}_\mathcal{R}(\Phi) = \varphi$.*

Proof. Combine the previous two theorems. $\qquad\square$

11. Main result

Combining our results with the completeness of filtered K-theory over accordion spaces, cf. Theorem 3.11, we get the following characterization of purely infinite graph algebras, and of purely infinite Cuntz-Krieger algebras.

Theorem 11.1. *Let X be an accordion space. The different versions of filtered K-theory introduced in this paper induce bijections between the following sets:*

- *stable isomorphism classes of tight, purely infinite graph algebras over X,*
- *isomorphism classes of stable Kirchberg X-algebras A of real rank zero with all simple subquotients in the bootstrap class satisfying that $\mathrm{K}_1\big(A(\{x\})\big)$ is free for all $x \in X$,*
- *isomorphism classes of countable, exact, real-rank-zero-like $\mathcal{NT}$-modules M with $M(\{x\}, 1)$ free for all $x \in X$,*
- *isomorphism classes of countable, exact, real-rank-zero-like, $\mathcal{ST}$-modules M with $M(\{x\}, 1)$ free for all $x \in X$,*
- *isomorphism classes of countable, exact $\mathcal{B}$-modules M with $M(x_1)$ free for all $x \in X$,*
- *isomorphism classes of countable, exact $\mathcal{R}$-modules M with $M(\overline{x}_1)$ free for all $x \in X$.*

Corollary 11.2. *Let X be an accordion space. The different versions of filtered K-theory introduced in this paper induce bijections between the following sets:*

- *isomorphism classes of tight, unital, purely infinite graph algebras over X,*

- *isomorphism classes of unital Kirchberg X-algebras A of real rank zero, with all simple subquotients in the bootstrap class such that, for all $x \in X$, the group $\mathrm{K}_1\big(A(\{x\})\big)$ is free and*

$$\mathrm{rank}\,\mathrm{K}_1\big(A(\{x\})\big) \leq \mathrm{rank}\,\mathrm{K}_0\big(A(\{x\})\big) < \infty,$$

- *isomorphism classes of countable, exact, real-rank-zero-like pointed $\mathcal{NT}$-modules M such that, for all $x \in X$, the group $M(\{x\}, 1)$ is free and*

$$\mathrm{rank}(M(\{x\}, 1)) \leq \mathrm{rank}(M(\{x\}, 0)) < \infty,$$

- *isomorphism classes of countable, exact, real-rank-zero-like pointed $\mathcal{ST}$-modules M such that, for all $x \in X$, the group $M(\{x\}, 1)$ is free and*

$$\mathrm{rank}(M(\{x\}, 1)) \leq \mathrm{rank}(M(\{x\}, 0)) < \infty,$$

- *isomorphism classes of countable, exact pointed $\mathcal{B}$-modules M such that, for all $x \in X$, the group $M(x_1)$ is free and*

$$\mathrm{rank}(M(x_1)) \leq \mathrm{rank}(\mathrm{coker}(\bigoplus_{y \to x} M(\widetilde{y}_0) \to M(\widetilde{x}_0))) < \infty,$$

- *isomorphism classes of countable, exact pointed $\mathcal{R}$-modules M such that, for all $x \in X$, the group $M(\overline{x}_1)$ is free and*

$$\mathrm{rank}(M(x_1)) \leq \mathrm{rank}(\mathrm{coker}(M(\widetilde{\partial}x_0) \to M(\widetilde{x}_0))) < \infty.$$

Corollary 11.3. *Let X be an accordion space. The different versions of filtered K-theory introduced in this paper induce bijections between the following sets:*

- *isomorphism classes of tight Cuntz Krieger algebras over X,*
- *isomorphism classes of unital Kirchberg X-algebras A of real rank zero, with all simple subquotients in the bootstrap class such that, for all $x \in X$, the group $\mathrm{K}_1\big(A(\{x\})\big)$ is free and*

$$\mathrm{rank}\,\mathrm{K}_1\big(A(\{x\})\big) = \mathrm{rank}\,\mathrm{K}_0\big(A(\{x\})\big) < \infty,$$

- *isomorphism classes of countable, exact, real-rank-zero-like pointed $\mathcal{NT}$-modules M such that, for all $x \in X$, the group $M(\{x\}, 1)$ is free and*

$$\mathrm{rank}(M(\{x\}, 1)) = \mathrm{rank}(M(\{x\}, 0)) < \infty,$$

- *isomorphism classes of countable, exact, real-rank-zero-like pointed $\mathcal{ST}$-modules M such that, for all $x \in X$, the group $M(\{x\}, 1)$ is free and*

$$\mathrm{rank}(M(\{x\}, 1)) = \mathrm{rank}(M(\{x\}, 0)) < \infty,$$

- *isomorphism classes of countable, exact pointed $\mathcal{B}$-modules M such that, for all $x \in X$, the group $M(x_1)$ is free and*

$$\mathrm{rank}(M(x_1)) = \mathrm{rank}(\mathrm{coker}(\bigoplus_{y \to x} M(\widetilde{y}_0) \to M(\widetilde{x}_0))) < \infty,$$

- *isomorphism classes of countable, exact pointed $\mathcal{R}$-modules M such that, for all $x \in X$, the group $M(\overline{x}_1)$ is free and*

$$\mathrm{rank}(M(x_1)) = \mathrm{rank}(\mathrm{coker}(M(\widetilde{\partial}x_0) \to M(\widetilde{x}_0))) < \infty.$$

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Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark
  E-mail address: arklint@math.ku.dk

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark
  E-mail address: bentmann@math.ku.dk

Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kouhoku-ku, Yokohama 223-8522, Japan
  E-mail address: katsura@math.keio.ac.jp